Housing, Collateral Constraints and Fiscal Policy

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Abstract
Real estate plays a dual role for fiscal policy. While it serves as an important tax base, investments in real estate also can been observed to be effectively subsidized. Yet, the literature on optimal fiscal policy typically ignores real estate. The aim of this paper is to provide a theoretical framework for such an analysis. The paper develops a two-sector general equilibrium model with non-durable consumption goods and durable housing. I show in a representative agent framework that - corresponding to the principles of optimal capital income taxation - in comparison to a labor income tax a housing tax should be small in the long run due to the intertemporal distortions associated with taxation of durable housing. The analysis of optimal fiscal policy is then extended to a framework with capital market imperfections, i.e. with lenders and collateral constrained borrowers for whom real estate serves as collateral. I show that optimal fiscal policy should disburden constrained agents by subsidizing their housing, while taxing housing of the unconstrained ones and labor income.

JEL classification: E44, E62, H21, R21, R38

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1 Introduction

Since the recent crisis economic literature on housing is booming. Before, this strand of economic literature was not widespread since this topic was neglected and regarded as a niche that is not of elemental importance for the macroeconomy. This view however changed significantly after the recent crisis where “housing [...] played a central role in precipitating” it, according to Ben Bernanke (2008).

While with this change in the view research on real estate economics grew rapidly, papers on optimal fiscal policy concerning housing are still rare. One might think that this issue is not that relevant to be addressed by economic research but looking at housing policies all around the world calls for theoretical as well as empirical research in this field.

On the one hand, housing wealth constitutes a substantial part of national wealth in many developed countries (see Iacoviello (2009)) and thus builds an important tax base. In the US about half of household wealth consists of housing wealth and in the first quarter of 2012 property taxes (referring to real as well as personal property) accounted for almost 35% of total government (state and local) revenues (Census (2012)). On the other hand, about 5 million households and nearly 10 million people made use of housing subsidies in 2009 in the US, according to the Department of Housing and Urban Development. Furthermore, there are several types of housing tax/subsidy policies applied all around the world that differ across countries (see e.g. Lam (2011) or IMF (2011)). In contrast to these widespread practices less research on these issues was conducted in the past.
The aim of this paper is to provide a theoretical framework to get theory-based results for the design of optimal fiscal policy with respect to housing in the long run and thus to investigate whether existing policies are justified in terms of welfare maximization. Therefore, I will consider two different models where the main difference is the use of housing as collateral in the second one.

In the first policy analysis, I consider a representative agent framework with a two-sector production side that integrates this work into renowned literature and serves as benchmark for the further analysis. In this benchmark model, the representative household derives utility from consumption, leisure and housing, while housing has no further use, especially no collateral function. On the production side of the economy there are two sectors, the producer of non-durable consumption goods and the one of durable housing. In this model, I consider two taxes, a housing property tax and a labor income tax in order to analyze optimal fiscal policy in the long run, where all taxes can also become negative, i.e. subsidies.

Along the lines of the literature on capital income taxation (Chamley (1986) and Judd (1985)), the results suggest that the housing tax should be quite small in modulus in the long run. Due to the characteristic that housing is a durable good and can be accumulated over time the tax rate is small since the housing tax incorporates intertemporal distortions. In contrast to the famous capital income tax result however the housing tax here can become positive, negative or exactly zero depending on the utility function. With an additively separable CRRA utility function the latter result emerges for the case that the intertemporal elasticities of
housing and consumption are equal. If the elasticity of housing is smaller than the one of consumption, the tax rate is positive and otherwise negative. In accordance with optimal taxation theory, housing should be taxed at a higher rate for a lower elasticity compared to consumption that is not taxed at all. Besides, as a tax smoothing result the tax rate on labor income is – even outside the steady state – constant and positive.

With a plausible calibration of the parameters and the governmental variables (government spending rate at 40% and government borrowing at 60%), the resulting tax rate on labor income is 34%, which is in the range of the effective tax rate estimates by Mendoza et al. (1994) and the housing tax rate is 1.3%, which is quite low and hence compatible with the capital income tax result.

In the second policy analysis, I augment the benchmark model by dividing households into two groups, patient and impatient ones, to get borrowers and lenders in the economy. The new and additional function of housing in this scenario is that the borrowers can use their house as collateral for loans. More precisely, they are constrained in their borrowings by the value of their house multiplied by a factor $0 \leq m \leq 1$.

In this augmented model, I first consider the tax system from above with one housing tax and one labor income tax for all to compare the results with the ones of the benchmark model and then a tax system with one housing tax for each household type and one labor income tax for both types.

In the augmented model, there are several effects at work that influence optimal
tax rates. The different discounting of the two types, private and public borrowing as well as public spending play an important role. An interesting point is that the Ramsey planner has to choose an aggregate discount rate for the private sector. Since this aggregate discount rate is a weighted mean of the two households’ discount rates, the Ramsey planner makes errors. While the lender is more patient compared to the aggregate discount rate, the borrower is more impatient. Thus, in this formulation of the Ramsey problem discounting affects optimal tax rates strongly. Since my major interest lies in the investigation of the effects of private borrowing and the collateral constraint, I suggest an alternative formulation of the Ramsey problem such that the results are not influenced by discounting. Here, I can focus on the effects of the collateral constraint.

It turns out, that in the simple tax system private borrowing and thus the existence of a binding collateral constraint leads to a housing subsidy and a positive labor income tax in absence of government spending and borrowing. Furthermore, there is no need for taxation if no private borrowing takes place. In the other tax system the optimal housing tax for the borrower is negative, while the one for the lender is positive but close to zero and the labor income tax is positive. These results also embed the ones of the benchmark model since the unconstrained household faces a very small tax. The subsidy for the constrained one is financed by the positive labor income tax.

Lastly, the link of this paper to the literature on housing assistance programs (e.g. Jacob and Ludwig (2012)) has to be emphasized, since the results suggest
that housing of the constrained agents should be subsidized. The model gives a theoretical answer to the question raised by Jacob and Ludwig what effect housing assistance has on labor supplied by the subsidized households. Jacob and Ludwig say that "economic theory is ambiguous about the expected sign of any labor supply response". According to this work however, labor supply of these households will decrease, since labor income taxes will increase due to higher housing subsidies to cover government expenditures, which is in line with the empirical results of Jacob and Ludwig.

The last section gives a conclusion of the most important results of the paper.
2 Steady State Taxation of Housing

In this section, I present a representative agent general equilibrium framework, where in addition to consumption and leisure also housing delivers utility to the household, while it has no further use, especially no collateral function, which will be added in the next section. Like in Davis and Heathcote (2005) or Favilukis et al. (2012), I consider a two-sector production side, such that both housing supply and housing demand are modeled explicitly. There are two representative firms, one of which produces non-durable consumption goods and the other durable housing.

In this economy, I study optimal fiscal policy concerning housing in the long run. I consider two types of taxes, a housing property tax \( \tau^h \) and a labor income tax \( \tau^n \), where both taxes can also become negative, i.e. subsidies. The government is assumed to have expenditures \( g_t \) and no access to lump-sum taxes.

The aim of this section is to integrate this work into renowned optimal taxation literature and to provide a benchmark for the further analysis. Two famous results of this literature are first, that capital income taxes should be zero in the long run (Judd (1985) and Chamley (1986)) and second, that labor income taxes should be constant (Barro (1979)), the so-called tax smoothing result. In line with these famous results, I show in this benchmark model that in the steady state optimal taxation incurs a very small housing tax (close to zero) as a result of the durability of housing and a constant labor income tax also outside the steady state due to tax smoothing.
2.1 Setup
2.1.1 Households

The representative household maximizes the infinite sum of expected utility, that rises with consumption $c_t$ and housing $h_t$, while it decreases with the working time $n_t$. The household’s objective is given by

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, h_t, n_t),$$

with $u_c > 0$, $u_{cc} < 0$, $u_h > 0$, $u_{hh} < 0$, $u_n < 0$ and $u_{nn} > 0$,

where $0 < \beta < 1$ denotes the discount factor, subject to a budget constraint. I consider the following CRRA-specification of the utility function

$$u(c_t, h_t, 1 - n_t) = \frac{c_t^{1-\mu^c}}{1 - \mu^c} + \frac{h_t^{1-\mu^h}}{1 - \mu^h} - \frac{n_t^{1+\mu^n}}{1 + \mu^n},$$

(1)

where $\mu^{c,h}$ denotes the inverse of the intertemporal elasticity of substitution in consumption (housing) and $\mu^n$ the inverse of the Frisch elasticity of labour supply.

The household generates income from working $w_t n_t$ with $w_t$ being the real wage rate and the return of bond holdings $b_t^g$. Labor income is taxed at the rate $\tau_t^n$. Every period the household undertakes an adjustment in housing $h_t - (1 - \delta_h) h_{t-1}$ at the price of housing $p_{h,t}$ with $\delta_h$ being the depreciation rate of housing. The value of the housing stock owned by the household is taxed at the rate $\tau_t^h$. Thus I consider a housing property tax, that is proportional to the value of the current housing stock and is paid every period. The household’s further expenditures consist of consumption spending $c_t$ and investment in new bonds $\frac{b_{t+1}^g}{R_t}$. Thus the budget
constraint of the household is given by

\[ c_t + p_{h,t} \left[ (1 + \tau^h_t) h_t - (1 - \delta_t) h_{t-1} \right] + \frac{b^g_{t+1}}{R^g_t} = (1 - \tau^n_t) w_t n_t + b^g_t, \]

where \( b^g_t \) denotes government bonds with the relating gross interest rate \( R^g_t = 1 + r^g_t \).

### 2.1.2 Government

The government levies a flat-rate tax on labor income and a housing property tax and is able to borrow by issuing one-period bonds to finance an exogenous stream of government expenditures \( (g_t) \), that are only for consumption goods:

\[ g_t - \frac{b^g_{t+1}}{R^g_t} + b^g_t = \tau^h_t p_{h,t} h_t + \tau^n_t w_t n_t. \]

### 2.1.3 Firms

The production side of the economy is characterized by two sectors, one of which produces consumption goods \( y_c \) and the other housing \( y_h \). In both sectors there is a representative firm that produces its output with labor according to

\[ y_{c,t} = n_{c,t}, \]

\[ y_{h,t} = n_{h,t}, \]

where \( n_t = n_{c,t} + n_{h,t} \). Thus total labor supply is split between the two sectors.
2.1.4 Aggregate Resource Constraint

With \( w_t n_t = y_{c,t} + p_{h,t} y_{h,t} \) due to zero profits, the aggregate ressource constraint is given by

\[
ct + gt + p_{h,t} h_t = y_{c,t} + p_{h,t} y_{h,t} + (1 - \delta_h) p_{h,t-1} h_{t-1}
\]

or \( ct + gt = y_{c,t} \) and \( h_t = (1 - \delta_h) h_{t-1} + y_{h,t} \).

Thus the c-good is used for private or public consumption and the output of the housing sector increases the housing stock.

2.2 The Ramsey Problem

The first order conditions of the household are given by (with \( u_t^x = \frac{\partial u}{\partial x_t} \))

\[
u_t^h + E_t \beta u_{t+1}^c (1 - \delta_h) p_{h,t+1} = u_t^c (1 + \tau_t^h) p_{h,t}
\]  

(2)

\[
u_t^n = -u_t^c w_t (1 - \tau_t^c)
\]  

(3)

\[
u_t^c = E_t u_{t+1}^c \beta R_t^o
\]  

(4)

and the transversality condition on bonds holds

\[
\lim_{t \to \infty} E_t \beta^t u_t^c b_{t+1}^o R_t^o = 0.
\]

The last two equations are a standard labor supply function and an intertemporal consumption Euler equation. The first equation describes the household’s demand for housing. Inserting (4) in (2) delivers the relationship between the marginal utilities of housing and consumption

\[
\frac{u_t^h}{u_t^c} = (1 + \tau_t^h) p_{h,t} - \frac{(1 - \delta_h) R_t^o}{E_t p_{h,t+1}}.
\]  

(5)
Housing demand hence decreases with the tax rate on housing $\tau^h_t$, the current price $p_{h,t}$ and the interest rate $R^g_{t}$, while it increases with the expected next period price of housing $E_t p_{h,t+1}$.

The first order conditions of the firms lead to real wage rate $w_t = 1$ and to the price of housing $p_{h,t} = 1$.

Thus the aggregate resource constraint becomes with $y_{c,t} + p_{h,t} y_{h,t} = y_{c,t} + y_{h,t} = n_{c,t} + n_{h,t} = n_t$

$$c_t + g_t + h_t - (1 - \delta_h) h_{t-1} = n_t$$  \hfill (6)

The Ramsey problem is to maximize social welfare subject to the aggregate resource constraint (6) and the implementability constraint (17), which is derived in the appendix 5.1, and can be written as

$$J = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t, h_t, 1 - n_t) + \phi [u^c_t c_t + u^h_t h_t + u^n_t n_t] + \phi [(1 - \delta_h) p_{h,0} h_{-1} + b^g_0] \right\} + \rho_t [-c_t - g_t - h_t + n_t + (1 - \delta_h) h_{t-1}]$$

Insertion of the marginal utilities leads to

$$J = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t, h_t, 1 - n_t) + \phi \left\{ c_t^{1-\mu^c} + h_t^{1-\mu^h} - n_t^{1+\mu^n} \right\} + \rho_t [-c_t - g_t - h_t + n_t + (1 - \delta_h) h_{t-1}] \right\} + \phi [(1 - \delta_h) p_{h,0} h_{-1} + b^g_0]$$

The first order conditions of the Ramsey problem are given by

$$\frac{\partial J}{\partial c_t} = 0 \Rightarrow \rho_t = c_t^{-\mu^c} [1 + \phi (1 - \mu^c)]$$  \hfill (7)

$$\frac{\partial J}{\partial n_t} = 0 \Rightarrow \rho_t = n_t^{-\mu^n} [1 + \phi (1 + \mu^n)].$$  \hfill (8)

The shadow price of the aggregate resource constraint $\rho_t$ hence is equal to the
marginal utility of consumption (or labor respectively) multiplied by a term with the shadow price of the implementability constraint \( \phi \), which describes the distortions resulting from the taxes.

Equalizing (7) and (8) we get the optimal labor income tax

\[
(1 - \tau^m_t) = \left( \frac{m^m_t}{c_t} \right)^{1} \frac{[1 + \phi (1 - \mu^c)]}{[1 + \phi (1 + \mu^m)]}, \tag{9}
\]

The first order condition with respect to housing is given by

\[
\frac{\partial J}{\partial h_t} = 0 \Rightarrow h_t^{-\mu^h} + \phi (1 - \mu^h) h_t^{-\mu^h} - \rho_t + \beta E_t \rho_{t+1} (1 - \delta_h) = 0,
\]

which delivers the optimal tax rate on housing (see appendix 5.2)

\[
\tau^h_t = \phi (1 - \mu^c) - \phi (1 - \mu^h) \frac{h_t^{-\mu^h}}{c_t^{-\mu^c}} - \frac{\phi (1 - \mu^c) (1 - \delta_h)}{R_t^g}.
\tag{10}
\]

Thus the tax rate on housing increases with \( \mu^h \), i.e. it decreases with the intertemporal elasticity of substitution in housing \( \frac{1}{\mu^h} \). In line with the theory of optimal taxation, the housing tax should be the higher the lower is the elasticity. Furthermore, \( \tau^h_t \) increases in the depreciation rate \( \delta_h \) and the interest rate \( R_t^g \) as well as the current stock of housing \( h_t \) due to a higher tax base, while it decreases with current consumption \( c_t \).

### 2.2.1 Optimal Tax Rates

The optimal tax rate on labor income results from (9)

\[
\tau^m_t = \tau^n = 1 - \left[ \frac{1 + \phi (1 - \mu^c)}{1 + \phi (1 + \mu^m)} \right] = \frac{\phi (\mu^m + \mu^c)}{1 + \phi (1 + \mu^m)} > 0 \text{ for } \phi > 0,
\]
where we see that this tax rate is constant and non-negative since it only depends on the multiplier on the imlementability constraint \( \phi \geq 0 \) and the parameters \( \mu^c \) and \( \mu^h \). For the case \( \phi = 0 \), i.e. that there are no tax distortions, the resulting optimal labor income tax is \( \tau^n_t = 0 \).

As we can see in
\[
\frac{\partial \tau^n}{\partial \phi} = \frac{\mu^n + \mu^c}{[1 + \phi (1 + \mu^n)]^2} > 0,
\]
the labor income tax is increasing in \( \phi \) and is concave for \( \mu^c \geq 1 \) (see appendix 5.2).

The optimal tax rate on housing follows from (10) and is given by (see appendix 5.2)
\[
\tau^h_t = \frac{\phi}{1 - \phi (\mu^h - 1)} \left( \mu^h - \mu^c \right) \left( 1 - \frac{(1 - \delta_h)}{R^g_t} \right).
\]
In the steady state with \( R^g_t = \beta^{-1} \) the optimal long run housing tax rate is given by
\[
\tau^h = \frac{\phi \left( \mu^h - \mu^c \right)}{1 - \phi (\mu^h - 1)} (1 - \beta (1 - \delta_h)). \tag{11}
\]
For \( \phi = 0 \) also the resulting optimal housing tax rate is \( \tau^h_t = 0 \), since there are no tax distortions.

For \( \phi > 0 \) the sign of the tax rate, thus the question whether housing should be taxed or subsidized, depends on the parameters. The second factor in (11) can be neglected here since \( 1 - \beta (1 - \delta_h) \) is positive. Hence we only need to focus on the first factor.

Here the analysis has to be restricted to values of \( \phi < \phi^* = \frac{1}{\mu^h-1} \), since for larger values the second derivatives become positive resulting in minimums (see appendix 5.2). There are three cases:
1. For $\mu^c = \mu^h$ the resulting optimal tax on housing in the long run is zero. This is due to the fact that intertemporal elasticities of substitution in consumption and housing are equal. Since consumption is not taxed at all, this implies a zero housing tax equal to the zero consumption tax.

For the case of inequality between these elasticities we have to take into account the shadow value of the tax distortion.

2. If the elasticity of housing is smaller than the one of consumption, i.e. $\mu^c < \mu^h$, the optimal housing tax rate is positive.

3. For $\mu^c > \mu^h$ the optimal housing tax rate is negative since the elasticity of consumption is smaller than the one of housing and the tax on consumption is zero.

The housing tax rate is increasing in $\phi$ for case 2 and decreasing for case 3, as

$$\frac{\partial \tau^h}{\partial \phi} = \frac{(\mu^h - \mu^c)}{[1 - \phi (\mu^h - 1)]^2 [1 - \beta (1 - \delta_h)]}$$

shows. Furthermore, $\tau^h$ is convex for case 2 and $\mu^h \geq 1$ (see appendix 5.2).

2.2.2 Numerical Results

In order to get an overview of the resulting tax rates of the model, we now give some numerical results for the steady state. The parameter values are set such that

$$\beta = 0.99, \quad \mu^h = 3.7, \quad \mu^c = 2, \quad \mu^n = 0.2, \quad \delta_h = 0.01, \quad (12)$$

since this calibration delivered the best approximation for macroeconomic variables in an earlier version of this paper working with a more elaborate version of this
model. For illustration figure 1 plots the optimal tax rates $\tau^h$ and $\tau^n$ as a function of $\phi$ for this calibration.

The additional governmental variables expenditures $g$ and borrowing $b^g$ are set to the values $g = 0.45$ and $b^g = 0.8$ such that their ratios to gdp are in the ballpark of the reported values in OECD (2008) $\frac{g}{y} \approx 0.36$ and $\frac{b^g}{y} \approx 0.63$. As the last column of table 1 shows the resulting tax rate on labor income is $\tau^n \approx 34\%$, which is in the range of the estimates of Mendoza et al. (1994) and the tax rate on housing is quite small ($\tau^h = 1.3\%$) along the lines of the literature on capital income taxation (e.g. Chamley (1986) and Judd (1985)). In contrast to their result however, the tax here is not exactly zero due to the fact that housing delivers utility and for this calibration more than consumption, leading to a positive, but small tax on housing. The second factor in (11) appears due to the fact that housing is a durable good, which does not fully depreciate within a period. If we had modeled housing as another non-durable consumption good, then this factor would disappear and the tax rate on housing would be in the range of the tax rate on labor income. This would also happen, if we assumed full depreciation ($\delta_h = 1$). To get an idea about the magnitude of the effect of the second factor in (11) we put in the parameter values $(1 - \beta (1 - \delta_h)) = 1 - 0.99^2 = 0.0199 \approx \frac{1}{50}$. Thus, the feature of being durable reduces the tax on housing in this calibration by a factor of about $\frac{1}{50}$ or, put differently, by 98%. Even if $g$ is so high that the labor income tax is greater than 50%, the housing tax remains quite small at about 1% as can be seen in figure 1 for $\phi > 0.3$. 

14
To get a further insight into the model, we can also use the government variables for policy experiments. Table 1 summarizes the resulting allocations and tax rates for different values of $g$ and $b^g$.

If there are no government expenditures $g = 0$ (and no borrowing $b^g = 0$) the implementability constraint becomes non-binding and the multiplier on it, $\phi$, is zero, such that the resulting taxes are zero. Government expenditures which are waste in our model reduce social welfare in crowding out private consumption of final goods and rising working time. The role of borrowing is slightly different, since it is not only a disbursement of the government but also an income of the household side. The
Table 1: Numerical Results I.

<table>
<thead>
<tr>
<th></th>
<th>$g = 0$</th>
<th>$g = 0$</th>
<th>$g = 0.1$</th>
<th>$g = 0.3$</th>
<th>$g = 0.4$</th>
<th>$g = 0.45$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$b^g = 0$</td>
<td>$b^g = 0.8$</td>
<td>$b^g = 0$</td>
<td>$b^g = 0$</td>
<td>$b^g = 0$</td>
<td>$b^g = 0.8$</td>
</tr>
<tr>
<td>$c$</td>
<td>0.9974</td>
<td>0.9940</td>
<td>0.9476</td>
<td>0.8571</td>
<td>0.8167</td>
<td>0.7942</td>
</tr>
<tr>
<td>$h$</td>
<td>2.8785</td>
<td>2.8686</td>
<td>2.7412</td>
<td>2.4583</td>
<td>2.3119</td>
<td>2.2208</td>
</tr>
<tr>
<td>$n$</td>
<td>1.0262</td>
<td>1.0227</td>
<td>1.0751</td>
<td>1.1817</td>
<td>1.2398</td>
<td>1.2664</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.0034</td>
<td>0.0425</td>
<td>0.1258</td>
<td>0.1655</td>
<td>0.1889</td>
<td></td>
</tr>
<tr>
<td>$\tau^h$</td>
<td>0</td>
<td>0.0001</td>
<td>0.0016</td>
<td>0.0064</td>
<td>0.0101</td>
<td>0.0130</td>
</tr>
<tr>
<td>$\tau^n$</td>
<td>0</td>
<td>0.0075</td>
<td>0.0889</td>
<td>0.2405</td>
<td>0.3038</td>
<td>0.3388</td>
</tr>
<tr>
<td>$u$</td>
<td>-1.8835</td>
<td>-1.8836</td>
<td>-1.9885</td>
<td>-2.2176</td>
<td>-2.3416</td>
<td>-2.4085</td>
</tr>
</tbody>
</table>

The distorting effect of $b^g$ is much smaller since $b^g$ is multiplied by small number $(1 - \beta)$ reflecting the fact that only interest payments are made. Furthermore, in the steady state borrowing acts like a redistribution from labor income to interest income of the same household. Considering the steady state government budget constraint makes this point clear: $g + (1 - \beta) b = \tau^h h + \tau^n n$. With the values in column two of table 1 we get $(1 - \beta) b \approx \tau^n n$. This point becomes more interesting in the augmented model, where we will consider borrowers and lenders and a redistribution between these two types of households.

Here, the only purpose of taxation is to cover the government expenditures, since there are no distortions. If government expenditures are zero, there is no need for taxation. In the augmented model, however, due to a preexisting distortion, there will be another purpose, which has to be served by taxation, namely redistribution between lenders and borrowers.
3 Collateral Constraints and Taxation of Housing

In this section, I analyze the effects of introducing capital market imperfections, i.e. a binding borrowing constraint, into the benchmark model on the resulting allocation and on optimal fiscal policy. I follow Kiyotaki and Moore (1997), who pioneered the models with two types of agents, lenders and borrowers, resulting from different time preference rates. Housing was embedded in this framework by Iacoviello (2005) with real estate delivering utility and being used as collateral for loans.

I augment the benchmark model in that way that I divide households into two groups, patient and impatient ones, to get lenders and borrowers in the economy. The borrowers use their housing wealth as collateral when they want to borrow. They are constrained in their borrowings by the value of their housing stock times a constant \( 0 \leq m \leq 1 \), which means that only a fraction of housing wealth is accepted as collateral. The households are different with respect to their discounting of future utility. There are patient households with a discount factor \( \beta \) and impatient households with \( \beta' < \beta \). This implies that in equilibrium the patient households become lenders and the impatient ones become borrowers.

To see this, we add private lending among the two types of households into the benchmark model. Denote that a household can lend [borrow] an amount \([-]b_t\) in period \( t - 1 \) which delivers him in period \( t \) a payback of \([-](1 + r_{t-1})b_t\). Put differently you lend \( \frac{b_t}{1 + r_{t-1}} \) and get back \( b_t \), where \( r_{t-1} \) is the real interest rate on loans between period \( t - 1 \) and \( t \).
In this new economy with imperfect capital markets, where a share of households faces a binding collateral constraint, I study optimal fiscal policy. As Hubbard and Judd (1986) as well as Chamley (2001) show, the zero capital income tax result does no more hold in the presence of liquidity constraints. Thus, one aim of this study is to investigate whether optimal fiscal policy in this new environment with borrowing constraints differs from the one in the benchmark economy. Another question I will explore is, whether fiscal policy can improve the equilibrium outcome of the augmented model with preexisting distortions.

For comparison, I first consider the tax system from above (shortly, $FP1$): two types of taxes, a housing property tax ($\tau^h$) and a labor income tax ($\tau^n$), which are applied to both types. Then, I split the housing tax and consider a tax system (shortly, $FP2$) where the tax rate on housing of the constrained household ($\tau^{h,b}$) can differ from the one of the unconstrained household ($\tau^{h,l}$). This will alter the results, as we well see. The government is assumed to have expenditures ($g_t$) and no access to lump-sum taxes, while borrowing is permitted.

3.1 Setup
3.1.1 Households

Households are divided into two types, patient (lenders) and impatient (borrowers) ones. They differ in their discount factors $1 > \beta > \beta' > 0$. Henceforth, variables without (with) a prime belong to the patient (impatient) households, while aggregate variables are denoted with a superscript $T$ (e.g. $c^T_t$, for total consumption). The share of the patient households is $s^l$. Since the impatient households can only borrow
up to certain fraction of their housing wealth, they will face a binding collateral constraint. Therefore the impatient households are also called constrained and the patient ones unconstrained households.

Both types of households maximize the infinite sum of expected utility from consumption $c_t^{(i)}$, labor $n_t^{(i)}$ and housing $h_t^{(i)}$. The household’s objective is given by

$$
\max E_0 \sum_{t=0}^{\infty} \beta^{(i)t} u(c_t^{(i)}, h_t^{(i)}, n_t^{(i)}),
$$

with $0 < \beta' < \beta < 1$, where $\beta$ ($\beta'$) denotes the discount factor of the unconstrained (constrained) household, subject to a budget constraint. As before, we consider the CRRA utility function (1).

**Unconstrained Households**

The budget constraint of the patient households reads

$$
c_t + (1 + \tau_t^h) p_{h,t} h_t + \frac{b_{t+1}^g}{R_t^g} - b_{t+1}^g + \frac{b_t}{R_t} - b_t = (1 - \tau_t^n) w_t n_t + (1 - \delta_h)p_{h,t} h_{t-1},
$$

where $\tau_t^h$ denotes a housing property tax, $\tau_t^n$ a labor income tax. Government bonds are denoted with $b_t^g$ and private lending with $b_t$, while $R_t^g = 1 + r_t^g$ and $R_t = 1 + r_t$ are the corresponding gross interest rates. The patient household will be the (private) lender in equilibrium.

**Constrained Households**
The budget constraint of the impatient households reads

\[ c_t' + (1 + \tau_t^h)p_{h,t}h_t' + \frac{b_{t+1}'}{R_t} - b_t' = (1 - \tau_t^h)w_t' + (1 - \delta_h)p_{h,t}h_{t-1}'. \]

Lending/borrowing between impatient households and the government is ruled out. This type will be the private borrower in equilibrium and has a limit on his private borrowings given by a fraction of his expected end of period housing wealth

\[ b_{t+1}' \geq -mE_t[p_{h,t+1}h_t']. \]

### 3.1.2 Government

The government levies a flat-rate tax on labor income and a housing property tax and is able to borrow by issuing one-period bonds to finance an exogenous stream of government expenditures \( g_t \):

\[ g_t - \frac{b_{t+1}'}{R_t} + b_t' = \tau_t^p p_{h,t}h_T + \tau_t^n w_T, \tag{13} \]

where \( h_T = s'h_t + (1 - s')h_t' \) and \( n_T = s'n_t + (1 - s')n_t' \).

#### 3.1.3 Firms

In both sectors there is a representative firm that produces its output with labor

\[ y_{c,t} = n_{c,t}^T \]
\[ y_{h,t} = n_{h,t}^T, \]

where total labor input in each sector is given by the weighted sum of labor input of the patient and impatient household in this sector \( n_{c,t}^T = s' n_{c,t} + (1 - s') n_{t,c} \) and
\[ n_{h,t}^T = s^t n_{h,t} + (1 - s^t) n_{h,t}' \]. On the other hand the labor supplied by the households is divided to the two firms \( n_t = n_{c,t} + n_{h,t} \) and \( n_t' = n_{c,t}' + n_{h,t}' \).

### 3.1.4 Aggregate Resource Constraint

The aggregate resource constraint is then given by (with \( w_t n_t^T = y_{c,t} + p_{h,t} y_{h,t} \) due to zero profits)

\[ c_t^T + g_t + p_{h,t} h_t^T = y_{c,t} + p_{h,t} y_{h,t} + (1 - \delta_h) p_{h,t} h_{t-1}^T. \]

### 3.2 The Ramsey Problem

#### 3.2.1 Households

The FOCs of the household side are given by the following equations. We again first consider the unconstrained and then the constrained household.

**Unconstrained Households**

The Lagrangian of the unconstrained household who chooses the values of \( c_t, h_t, n_t, b_{t+1}^g \) and \( b_{t+1} \) reads

\[
\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t, h_t, 1 - n_t) + \lambda_{1,t} \left[ (1 - \tau_t^h) w_t n_t + (1 - \delta_h) p_{h,t} h_{t-1} - \frac{b_{t+1}^g}{R^g} + b_t^g \right] \right. \\
\left. \quad - \frac{b_{t+1}}{R^t} + b_t - c_t - (1 + \tau_t^h) p_{h,t} h_t \right\}
\]

leading to the following FOCs

\[
h_t^{-\mu_h} = (1 + \tau_t^h) p_{h,t} c_t^{-\mu_c} - E_t \beta c_{t+1}^{-\mu_c} (1 - \delta_h) p_{h,t+1}
\]

\[
n_t^{-\mu_n} = (1 - \tau_t^h) w_t c_t^{-\mu_c}
\]

\[
c_t^{-\mu_c} = E_t \beta R_t^g c_{t+1}^{-\mu_c}
\]

\[
c_t^{-\mu_c} = E_t \beta R_t c_{t+1}^{-\mu_c}
\]

These are the standard FOCs as in section 2 for housing demand, labor supply and
the Euler equations with respect to public and private lending.

**Constrained Households**

The Lagrangian of the unconstrained household, choosing the values of \( c_t, h_t, n_t \) and \( b_{t+1} \), is given by

\[
\mathcal{L}' = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t, h_t, 1 - n_t') + \lambda_{t+1} \right. \\
&\quad \left. \left[ (1 - \tau_t^h) w_t n_t' + (1 - \delta_t) p_{h,t} h_{t-1} \right] \\
&\quad \left. - \frac{b_{t+1}'}{R_t} + b_t - c_t' - (1 + \tau_t^h) p_{h,t} h_t' \right\} \\
&\quad + \omega_t \left[ b_{t+1}' + m E_t [p_{h,t+1} h_t'] \right]
\]

leading to the following FOCs

\[
h_t'^{-\mu_h} = (1 + \tau_t^h) p_{h,t} c_t'^{-\mu_e} - E_t \beta^t c_t'^{-\mu_e} (1 - \delta_t) p_{h,t+1} + \omega_t m E_t p_{h,t+1}
\]

\[
n_t'^{\mu_n} = (1 - \tau_t^n) w_t c_t'^{-\mu_e}
\]

\[
\omega_t = \frac{c_t'^{-\mu_e} - \beta^t E_t c_{t+1}'^{-\mu_e} R_t}{R_t}
\]

The first equation describes housing demand of the constrained households. The additional summand \( \omega_t m E_t p_{h,t+1} \) comes from the borrowing constraint with \( \omega_t \) being the shadow price of this constraint. The second equation is the standard labor supply function of the constrained household.

Furthermore, the transversality conditions

\[
\lim_{t \to \infty} E_t \beta^t u_t c_t' \frac{-b_{t+1}'}{R_t^g} = 0
\]

and

\[
\lim_{t \to \infty} E_t \beta^t u_t' b_{t+1}' \frac{R_t'}{R_t} = 0,
\]

must hold, from which the latter is redundant due to the collateral constraint that is more restrictive.
3.2.2 Government

In order to get the intertemporal government budget constraint, which we derive to rule out a ponzi game of the government we write (13) for \( t+1 \), solve for

\[
b_{t+1}^g = \tau_t^h p_{h,t+1} h_{t+1}^T + \tau_t^w w_{t+1} n_{t+1}^T - g_{t+1} + \frac{b_{t+2}^g}{R_{t+1}}
\]

and insert this in the one for \( t \)

\[
g_t - \frac{1}{R_t^g} \left[ \tau_t^h p_{h,t} h_t^T + \tau_t^w w_t n_t^T - g_t + \frac{b_{t+2}^g}{R_{t+1}} \right] + b_t^g = \tau_t^h p_{h,t} h_t^T + \tau_t^w w_t n_t^T,
\]

or rewritten

\[
g_t + \frac{g_{t+1}}{R_t^g} - \frac{b_{t+2}^g}{R_t^g R_{t+1}^g} + b_t^g = \tau_t^h p_{h,t} h_t^T + \frac{\tau_t^h p_{h,t+1} h_{t+1}^T}{R_t^g} + \tau_t^w w_t n_t^T + \frac{\tau_t^w w_{t+1} n_{t+1}^T}{R_t^g}.
\]

Iterating on this we get with the transversality condition on government debt the intertemporal government budget constraint

\[
\sum_{t=0}^{\infty} \left( \prod_{i=0}^{t-1} (R_i^g)^{-1} \right) g_t + b_0^g = \sum_{t=0}^{\infty} \left( \prod_{i=0}^{t-1} (R_i^g)^{-1} \right) \tau_t^h p_{h,t} h_t^T + \sum_{t=0}^{\infty} \left( \prod_{i=0}^{t-1} (R_i^g)^{-1} \right) \tau_t^w w_t n_t^T
\]

\[
\Leftrightarrow \sum_{t=0}^{\infty} \left( \prod_{i=0}^{t-1} (R_i^g)^{-1} \right) \left[ g_t - \tau_t^h p_{h,t} h_t^T - \tau_t^w w_t n_t^T \right] + b_0^g = 0.
\]

3.2.3 Firms

The FOCs of the firms follow from maximization of profits \( \Pi_{c,t} = y_{c,t} - w_t n_{c,t}^T \) and

\[
\Pi_{h,t} = p_{h,t} y_{h,t} - w_t n_{h,t}^T
\]

and are given by \( w_t = 1 \) and \( p_{h,t} = 1 \).

3.2.4 Summary of Conditions

The constraints faced by the government, while choosing optimal tax rates, which are the FOCs of the unconstrained and constrained households, the aggregate ressource
constraint, the government budget constraint and the budget constraint of the unconstrained households, are summarized in the appendix (5.3).

3.2.5 Dual Approach

We have to solve this problem using the dual approach since we cannot derive an implementability constraint. The government chooses the values of $h_t, c_t, n_t, h'_t, c'_t, n'_t$ and the tax rates $\tau^h_t$ and $\tau^n_t$ in order to maximize social welfare, measured by the weighted sum of utility of the two types of households, subject to the household’s FOCs and the constraints. Following Monacelli (2008), I define the aggregate discount rate as $\tilde{\beta} = \beta^s \beta^{(1-s)}$ and use this one as the discount rate for the constraints, where the $\lambda_{t,i}$ denotes the Langrange multiplier on constraint $i$ in $t$. The Ramsey Problem thus reads

$$J = E_0 \sum_{t=0}^{\infty} \left\{ \beta^t s^t u(c_t, h_t, n_t) + \beta^{tt} (1 - s^t) u(c'_t, h'_t, n'_t) \right\}$$

$$+ \tilde{\beta} \lambda_{t,1} \left[ h_t^{-\mu^h} - (1 + \tau^h_t) c_t^{-\mu^c} - \beta (1 - \delta_h) E_t c_{t+1}^{-\mu^c} \right]$$

$$+ \tilde{\beta} \lambda_{t,2} \left[ n_t^{\mu^n} c_t^{-\mu^c} - 1 + \tau^n_t \right] + \tilde{\beta} \lambda_{t,3} \left[ n_t'^{\mu^n} c_t'^{-\mu^c} - 1 + \tau^n_t \right]$$

$$+ \tilde{\beta} \lambda_{t,4} \left[ h_t^{\mu^h} - (1 + \tau^h_t) c_t'^{-\mu^c} + \beta' (1 - \delta_h) E_t c_{t+1}^{\mu^c} - \beta' E_t c_{t+1}^{\mu^c} \right]$$

$$+ \tilde{\beta} \lambda_{t,5} \left[ -c_t' - (1 + \tau^h_t) h_t' + (1 - \tau^n_t) n_t' + (1 - \delta_h) h_{t-1}' \right]$$

$$+ \tilde{\beta} \lambda_{t,6} \left[ -s^t c_t - (1 - s^t) c'_t - g_t + s^t h_t - (1 - s^t) h_t' \right]$$

$$+ s^t n_t + (1 - s^t) n_t' + (1 - \delta_h) \left( s^t h_{t-1}' + (1 - s^t) h_{t-1}' \right)$$

$$+ \tilde{\beta} \lambda_{t,7} c_t^{-\mu^c} g_t - \tau^h_t \left( s^t h_t + (1 - s^t) h_t' \right) - \tau^n_t \left( s^t n_t + (1 - s^t) n_t' \right) + \beta^t \lambda_t b^g_0$$
The FOCs of the Ramsey problem and the solution for the steady state are derived in the appendix (5.4) and given by

\[
\begin{align*}
\lambda_1 c^{-\mu} + \lambda_4 c^{\mu} + \lambda_5 h' + \lambda_7 h^T &= 0 \\
\lambda_2 + \lambda_3 - \lambda_5 n' - \lambda_7 n^T &= 0
\end{align*}
\]

\[
\frac{s'c}{\mu c} + \lambda_1 \left[ 1 + \tau^h - \beta (1 - \delta_h) \right] + \lambda_2 (1 - \tau^n) c^{\mu c} + \lambda_4 mc^{\mu} c^{\mu} \left( \beta - \beta \right) \\
+ \lambda_5 mh' c^{\mu c} \left( \beta - \beta \right) - \lambda_6 s'c^{\mu c+1} \mu c - \lambda_7 c^{\mu c} \left[ g - \tau^h h^T - \tau^n n^T \right] = 0
\]

\[
\begin{align*}
h^{-\mu h} \left( 1 - \lambda_1 \frac{\mu h}{s'h} \right) + \lambda_6 \left[ \bar{\beta}(1 - \delta_h) - 1 \right] - \lambda_7 \tau^h &= 0 \\
- n^{\mu} + \lambda_2 \frac{\mu n (1 - \tau^n)}{s'n} + \lambda_6 - \lambda_7 n^h &= 0
\end{align*}
\]

\[
\frac{(1 - s') c'}{\mu c} + \lambda_3 (1 - \tau^n) c^{\mu c} + \lambda_4 \left[ 1 + \tau^h + m \beta - \bar{\beta}(1 - \delta_h + m) \right] \\
- \frac{\lambda_5}{\mu c^{\mu c-1}} - \frac{\lambda_6 (1 - s')}{\mu c^{\mu c-1}} = 0
\]

\[
\begin{align*}
h'^{-\mu h} \left( 1 - \lambda_4 \frac{\mu h}{(1 - s') h'} \right) + \lambda_5 \left[ \frac{\bar{\beta}(1 - \delta_h - m) - 1 - \tau^h + m \beta}{1 - s} \right] \\
+ \lambda_6 \left[ \bar{\beta}(1 - \delta_h) - 1 \right] - \lambda_7 \tau^h &= 0 \\
- n'^{\mu} + \lambda_3 \frac{\mu n (1 - \tau^n)}{(1 - s') n'} + \lambda_5 \frac{(1 - \tau^n)}{(1 - s')} + \lambda_6 - \lambda_7 n^n = 0.
\end{align*}
\]

### 3.3 Numerical Results

Again we look at some numerical results of the steady state to get an overview of the resulting optimal tax rates in the augmented model. The parameters are set to the values from section 2, summarized in (12): \( \beta = 0.99, \mu^h = 3.7, \mu^c = 2, \mu^n = 0.2, \delta_h = 0.01 \). Following Iacoviello (2005), the discount rate of the impatient household is set to \( \beta' = 0.95 \) and the collateralizable share of housing to \( m = 0.7 \), whereas the share of lenders in the population is set to \( s' = 0.5 \) as benchmark with
Table 2: Numerical Results IIa.

<table>
<thead>
<tr>
<th>Benchmark Model</th>
<th>Augmented Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g = 0$</td>
<td>$g = 0.45$</td>
</tr>
<tr>
<td>$b^g = 0$</td>
<td>$b^g = 0.8$</td>
</tr>
<tr>
<td>$c$</td>
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<td>$h$</td>
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<td>$n$</td>
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<tr>
<td>$c'$</td>
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<tr>
<td>$h'$</td>
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</tr>
<tr>
<td>$n'$</td>
<td>1.0349</td>
</tr>
<tr>
<td>$\tau^h$</td>
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<tr>
<td>$\tau^n$</td>
<td>0</td>
</tr>
<tr>
<td>$u$</td>
<td>-1.8835</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Augmented Model</th>
<th>Augmented Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g = 0$</td>
<td>$g = 0.45$</td>
</tr>
<tr>
<td>$b^g = 0$</td>
<td>$b^g = 0.8$</td>
</tr>
<tr>
<td>$c$</td>
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<tr>
<td>$h$</td>
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<td>$n$</td>
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</tr>
<tr>
<td>$c'$</td>
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</tr>
<tr>
<td>$h'$</td>
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</tr>
<tr>
<td>$n'$</td>
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</tr>
<tr>
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</tr>
<tr>
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<tr>
<td>$u$</td>
<td>-2.4085</td>
</tr>
</tbody>
</table>

equal shares.

Again we consider the resulting taxes for $g = 0.45$ and $b^g = 0.8$ as benchmark since they lead to the shares $\frac{g}{y} \approx 0.35$ and $\frac{b^g}{y} \approx 0.63$. The resulting tax rate on labor income is $\tau^n = 32\%$, which is again in the ballpark of the estimates of Mendoza et al. (1994) and the tax rate on housing is quite small ($\tau^h = 2.8\%$). In comparison to the benchmark model we get a slightly smaller labor and a slightly higher housing tax (see table 2, column 3 vs. 5). Also for $g = b^g = 0$ the results differ from the ones of the benchmark model. As table 2 shows the optimal tax rate for housing in this model is positive ($\tau^h = 0.89\%$), while the one for labor is negative with $\tau^n = -1.9\%$.

The emergence of these results is somewhat surprising since one would expect a housing subsidy rather than a labor income subsidy. There are several effects at work that lead to these results. We will now turn to these effects and in the subsequent section 3.4 we will regard a more elaborate tax system and will see that these results are also due to the simple tax system with one tax for all. To understand the workings of the model, we first analyze the effects of the different
discount rates and population shares of the two types on optimal taxation. The examination of the effects of private borrowing, public borrowing and government spending rounds off the analysis.

**Discounting and Taxation**

As we have seen in table 2, also in a scenario where governmental variables are set to zero, the optimal tax rates differ from zero in the augmented model. The reason for these results is that the discount rates of each group (β vs. β’) differ from the aggregate one given by β’ < β = β's β'(1−s') < β. In order to focus solely on this effect, I shut down the borrowing channel and its effects on optimal taxes by setting m = 0. The parameter values for the results in table 3 are again given by (12), as well as g = b^9 = 0.

Denote that for β’ = 0.99 and m = 0 we get the same allocation and optimal tax rates as in section 2, second column of table 2, meaning that for both types having the same discount rate and no borrowing, we are in the benchmark model. As β’ becomes smaller relative to β, this drives a wedge between the two tax rates. The
reason for this is that in the Ramsey Problem the discount rate for the constraints was chosen as an aggregate for both types given by $\widetilde{\beta} = \beta^s / \beta^{(1-s')}$. With this aggregation the government makes errors from the private sector’s perspective. While it discounts too heavily from the lenders’ view ($\beta > \widetilde{\beta}$), it is too patient from the borrowers’ viewpoint ($\beta' < \widetilde{\beta}$). Using $\widetilde{\beta}$ the government wants the lender to consume more non-durables, less durables and work less and the borrower to consume less non-durables, more durables and work more compared to their optimal plans. Shortly, the government wants

$$
c \uparrow, \ h \downarrow, \ n \downarrow \text{ and } c' \downarrow, \ h' \uparrow, \ n' \uparrow
$$

and has two instruments to achieve these goals, $\tau^h$ and $\tau^n$, while it can only choose one tax rate freely and the other will then be given by the government budget constraint $\tau^h h^T = -\tau^n n^T$. The government faces a trade-off since with respect to lenders it should set $\tau^h > 0$ and $\tau^n > 0$, while it should set $\tau^h < 0$ and $\tau^n < 0$ with respect to borrowers. Since the borrower is the one who works more and the lender is the one who holds more housing, the optimal choice of the government being restricted through the budget constraint is: $\tau^h > 0$ and $\tau^n < 0$. Thus, the government subsidizes labor to get the households, especially the impatient ones, work more and taxes housing to reduce the housing stock, especially the lenders’ one. As we have seen, these results emerge from the aggregation of the discount rates which is not the main focus of this paper. In an alternative formulation of the Ramsey problem (see 3.5), I get the discount rate not to affect optimal tax rates and
Table 4: Population Shares and Taxation.

<table>
<thead>
<tr>
<th></th>
<th>$s^l = 0.4$</th>
<th>$s^l = 0.5$</th>
<th>$s^l = 0.6$</th>
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<tr>
<td>$h$</td>
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<td>2.5936</td>
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</tr>
<tr>
<td>$n$</td>
<td>1.0368</td>
<td>1.0361</td>
<td>1.0349</td>
</tr>
<tr>
<td>$c'$</td>
<td>1.0088</td>
<td>1.0085</td>
<td>1.0077</td>
</tr>
<tr>
<td>$h'$</td>
<td>2.0616</td>
<td>2.0655</td>
<td>2.0735</td>
</tr>
<tr>
<td>$n'$</td>
<td>1.0273</td>
<td>1.0267</td>
<td>1.0257</td>
</tr>
<tr>
<td>$\tau^h$</td>
<td>0.0105</td>
<td>0.0100</td>
<td>0.0089</td>
</tr>
<tr>
<td>$\tau^n$</td>
<td>-0.0231</td>
<td>-0.0225</td>
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<tr>
<td>$u$</td>
<td>-1.8992</td>
<td>-1.8970</td>
<td>-1.8947</td>
</tr>
</tbody>
</table>

thus can concentrate on the other effects, especially the ones resulting from private borrowing and the collateral constraint, which is the main interest of this paper.

**Population Shares and Taxation**

Now we take a look at how changes in the population shares in the neighbourhood of symmetry ($s^l = 0.5$) affect optimal tax rates with $m = g = b^g = 0$. The considerations here are closely related to the previous ones since the channel goes through the discount rates.

The higher the share of lenders in the economy the larger is their weight in the social welfare function. With respect to them the government should set $\tau^h > 0$ and $\tau^n > 0$, as we have seen above. Furthermore, the higher $s^l$ is, the closer is the aggregate discount rate $\tilde{\beta} = \beta^s \beta^{(1-s)}$ to the lenders’ one, $\beta$. Thus, a higher $s^l$ has same effects as a higher $\beta'$, namely lowering the housing tax $\tau^h$ and increasing the labor income tax $\tau^n$.

**Private Borrowing and Taxation**

We now examine the role of the parameter $m$, i.e. the fraction of collateralizable
Table 5: Private Borrowing and Taxation.

<table>
<thead>
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<tr>
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<tr>
<td>u</td>
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<td>-1.9057</td>
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housing, to analyze how private borrowing affects the allocation and the optimal tax rates in the long run. With the benchmark calibration from (12) and \(\beta' = 0.95\) as well as \(g = b^g = 0\) to abstract from public borrowing and spending, we get the results in table 5.

For higher \(m\) the constrained household is worse off in the steady state since he borrowed more and therefore has to pay higher interest payments leading to lower values of consumption of non-durables and durables as well as less working time. A higher \(m\) also reduces \(h'\) since the household needs less housing to borrow the same amount. The lender is better off for a higher \(m\) that reduces his working time and increases his housing demand due to higher interest payments from the borrower. In the aggregate, however, the effect of a higher \(m\) on total housing \(h^T\) is negative since the negative effect on \(h'\) is stronger than the positive effect on \(h\). In order not do depress the housing even more, the housing tax is lower and due to this the subsidy on labor income is also lower.
**Government Spending, Public Borrowing and Taxation**

To see how they influence optimal tax rates, we use the government variables for policy experiments. Table 6 summarizes the resulting allocations and optimal tax rates for different values of $g$ and $b^g$ with the benchmark calibration (12) and $\beta' = 0.95$, as well as $m = 0$ in order to abstract from private borrowing.

Here again the different roles of government spending and government borrowing become evident. A rise in government spending $g$ is bad for both types of households, since it crowds out consumption of durables and non-durables and rises both tax rates forcing the households to work more. These effects are all welfare reducing. The effect of government borrowing is much smaller compared to spending since only interest payments on them are relevant in the steady state. Furthermore, the role of government borrowing is different because only the patient household lends to the government and thus gets the interest payments of $(1 - \beta) b^g$, what acts like a redistribution to the lender making him work less (see table 6: $n = 1.0361$ for $b^g = 0$ vs. $n = 1.0335$ for $b^g = 0.3$).

---

Table 6: Government Spending and Taxation.

<table>
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<tr>
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<td>2.3553</td>
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<td>1.0335</td>
<td>1.1386</td>
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<td>0.9119</td>
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<td>1.9197</td>
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<td>1.0276</td>
<td>1.1270</td>
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<td>0.0150</td>
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<td>$u$</td>
<td>-1.8970</td>
<td>-1.8971</td>
<td>-2.1152</td>
</tr>
</tbody>
</table>
At a higher borrowing level the government has to raise taxes due to higher interest payments. Since the lender will work less because of higher interest payments, the labor income tax will go up while the housing tax will remain at its level in order not to worsen the situation of the constrained household, because rising housing taxes would hit the impatient household more than the patient one due to the collateral constraint. Now we take a look at the case where both, government spending and borrowing, are zero.

**Conclusion**

Since in the benchmark model the only distortion was resulting from taxation due to wasteful government expenditures, eliminating them \((g = 0, b^g = 0)\) eliminated the tax distortions \((\phi = \tau^h = \tau^n = 0)\) and led to the welfare maximizing allocation. Here however we see that \(g = b^g = 0\) does not lead to \(\tau^h = \tau^n = 0\). One reason for this is the different discounting as we have seen above in absence of private borrowing. If private borrowing takes place the collateral constraint as a preexisting distortion becomes relevant.

The main focus of this paper is to analyze the effects of this constraint. Thus the considerations with \(g = b^g = 0\) and \(m > 0\) are the most relevant results focusing on the effects of this constraint. While in the benchmark model the only purpose of taxation was meeting the expenses of the government, here another purpose for taxation emerges. It is the redistribution from the unconstrained household to the constrained ones in order to eliminate distortions resulting from the collateral constraints.
Whether we can improve even more or not we will see in the next subsection where we implement a simple transfer system and analyse its welfare effects.

### 3.3.1 Redistribution and Welfare Maximum

To find the welfare maximizing allocation we add a transfer between the two types of households. A transfer that makes the collateral constraint non-binding ($\lambda_5 = 0$) will lead to this allocation. As in Monacelli et al. (2008) we consider a pure redistribution from the lenders $TR_t$ to the borrowers $TR'_t$ given by

$$TR_t = -\varepsilon_t \quad \text{and} \quad TR'_t = \frac{s^l}{1 - s^l} \varepsilon_t \quad \text{with} \quad \varepsilon_t > 0$$

holding everything else constant. The numerical analysis for the parameters above with $g = b^g = 0$ and $m = 0.7$ and $s^l = 0.5$ leads to an optimal transfer in the amount of $TR_t = -0.0097$ from the lenders to the borrowers. The resulting allocation given in table 7 leads to a higher value of social welfare ($-1.90301$ vs. $-1.90304$).

The considerations above that there are counteracting effects of the housing tax rate with respect to the households and the fact that this transfer can improve
social welfare suggest that we can do better in the augmented model with a more elaborate tax system. As we can see in table 7, the two types of households consume roughly the same and only differ a little in their working times. So the tax system had success in equalizing them along these dimension, but it led to big differences in housing. Since we only have one tax for housing and this tax has to be set to different values from the different household perspectives, this difference occurs.

Obviously, the point where the tax system has to be changed is the housing tax, which is the object of the next section where the housing tax on the borrowers can differ from the one on the lenders.

3.4 Alternative Policies

In this section, we consider an alternative policy that is able to increase social welfare with respect to the previous tax system. More precisely, we dismiss the idea of one tax on housing for both types $\tau^h_t$ and embed a tax system with different tax rates on housing: $\tau_{h,l}^t$ for the lender and $\tau_{h,b}^t$ for the borrower. This tax system (from now on $FP2$) can be thought of as a short-cut for housing tax systems that prefer a certain group in population (here the constrained ones) that are apparent all over the world (see e.g. Lam (2011)). As a result, this section provides a theoretic argument in favor of these programs. The Langrangian and the steady state for this problem are given in Appendix 5.5.

3.4.1 Numerical Results

We use the benchmark calibration reported in (12). First, we compare the results of this tax system ($FP2$) with the former one ($FP1$) and the benchmark model with
As table 8 shows $FP_2$ outperforms $FP_1$ with respect to social welfare and is consistent with the intuition that the housing tax rate should be negative or at least lower for the borrower compared to the lender, since he faces the collateral constraint.

Without government spending and borrowing the resulting optimal housing tax for the borrower is negative, while it is positive for the lender. Also for realistic values of $g$ and $b^g$ the tax system prefers the constrained household since $\tau^{h,b} < \tau^{h,l}$. The resulting tax rates for the benchmark calibration with $g = 0.45$ and $b^g = 0.8$ are $\tau^n = 33\%$ (consistent with Mendoza et al. (1994)), $\tau^{h,l} = 3.8\%$ and $\tau^{h,b} = 0.1\%$, i.e. the tax rate on housing for the borrower is no more negative but still much smaller than the one for the lender.

$FP_2$ always leads to a higher value of social welfare compared to $FP_1$. The reason for this is that now the ones are subsidized, who have to be subsidized.
Table 9: Discounting, Population Shares and Taxation.

<table>
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<th>$\beta' = 0.95$</th>
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<td>$s' = 0.5$</td>
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</tr>
<tr>
<td>$h'$</td>
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<td>2.3698</td>
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<tr>
<td>$n'$</td>
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<td>0.9818</td>
</tr>
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<td>$\tau^{h,b}$</td>
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<td>-0.0182</td>
</tr>
<tr>
<td>$u$</td>
<td>-1.8856</td>
<td>-1.8929</td>
</tr>
</tbody>
</table>

to increase welfare, namely the constrained households, while the housing tax on unconstrained households is positive but small. Comparing the housing tax for borrowers and lenders with the housing tax in the former system ($FP_1$) we get the following relationship $\tau^{h,l}_{FP_2} > \tau^{h}_{FP_1} > \tau^{h,b}_{FP_2}$. Splitting the tax rate on housing drives a wedge between the two types’ tax rates. The smaller rate for the constrained household rises also the tax rate on labor to serve government spending: $\tau^{n}_{FP_2} > \tau^{n}_{FP_1}$.

We now again focus on the underlying effects leading to these results.

**Discounting, Population Shares and Taxation**

The results in table 9 are based on (12) and $m = g = b^g = 0$.

We see that now again discounting drives a wedge between the three taxes. For a smaller $\beta'$, the lenders’ housing tax goes up, while the borrowers’ one and the labor income tax go down. As we have seen before the government should set $\tau^{h,l} > 0$ and $\tau^{n} > 0$ with respect to lenders and $\tau^{h,b} < 0$ and $\tau^{n} < 0$ with respect to borrowers. Since now the government is able to distinguish between lenders and
borrowers while taxing their housing, it sets $\tau^{h,l} > 0$ to reduce housing of the lender, $\tau^{h,b} < 0$ to increase housing of the borrower and $\tau^n$ very small (in modulus) due to the conflicting situation in setting $\tau^n$.

An increase in $s^l$ reduces $\tau^{h,l}$ since now the lenders become more important regarding social welfare. But also $\tau^{h,b}$ decreases since there are less borrowers who are subsidized and more lenders who are taxed.

**Borrowing, Government Spending and Taxation**

We now look at the effects of private borrowing, public borrowing and government spending. The results in table 10 are based on (12) and $\beta' = 0.95$.

Comparing columns 2 and 3 of table 10, shows that private borrowing gives rise to subsidize the constrained households even stronger ($\tau^{h,b} = -4.3\%$). To finance this, the labor income tax has to increase leading both household to work less. The role of government borrowing and spending are the same as before. Government spending crowds out consumption of both goods and increases all tax rates forcing
the households to work more. All of these effect are welfare reducing.

These results give also a theoretical answer to the question what effect housing assistance has on labor supply of the subsidized households. Jacob and Ludwig (2012) give an overview of previous results concerning this issue and say that "economic theory is ambiguous about the expected sign of any labor supply response". The sign resulting from their empirical study is negative, saying that housing assistance programs lower labor supply by these households. My results are in line with these ones as we can see in table 10. Since higher housing subsidies come along with higher labor income taxes the constrained households will reduce labor supply.

### 3.5 Alternative Formulation of the Ramsey Problem

In the previous analyses we have seen that the discount rates played an important role for optimal taxation. Since the major interest of this paper lies in the investigation of the effects of the other factors, especially private borrowing and the collateral constraint, I want discounting per se not to affect optimal tax rates. Therefore, I suggest an alternative formulation of the Ramsey problem that leads to results that are not biased by discounting and solely reflect the effects I am interested in.

I do the steps from above in order to compare the resulting tax rates and to make clear how the two formulations lead to different results. First I consider the simple tax system ($FP_1$) and then the other ($FP_2$).
The Lagrangian for this problem reads

\[
J = E_0 \sum_{t=0}^{\infty} \begin{pmatrix}
\beta^t s^t u(c_t, h_t, n_t) + \beta'^t (1 - s^t) u(c'_t, h'_t, n'_t) \\
+ (\beta^*)^t \lambda_{t,1} \left[ h_t^{\mu h} - (1 + \tau^h_t) c_t^{\mu c} + \beta (1 - \delta_h) E_t c_{t+1}^{\mu c} \right]
+ (\beta^*)^t \lambda_{t,2} \left[ n_t^{\mu c} - 1 + \tau^h_t \right] + \beta^t \lambda_{t,3} \left[ n_t^{\mu c} - 1 + \tau^h_t \right]
+ (\beta^*)^t \lambda_{t,4} \left[ h_t^{\mu h} - (1 + \tau^h_t) c_t^{\mu c} + \beta' (1 - \delta_h) E_t c_{t+1}^{\mu c} \right]
- m \left( c_t^{\mu c} c_{t}^{\mu c} - E_t c_{t+1}^{\mu c} \right)
\left( c_t^{\mu c} + \beta' E_t c_{t+1}^{\mu c} \right)
+ \lambda_{t,5} \left[ -c'_t - (1+\tau^h_t) h'_t + (1 - \tau^h_t) n'_t + (1 - \delta_h) h'_{t-1} \right]
+ \lambda_{t,6} \left[ m h'_t \beta E_t c_{t+1}^{\mu c} - m h'_t \right]
\end{pmatrix} e^{\lambda_{t,7} \frac{c_t^{c}}{c_0^{\mu c}}} \left[ g_t + \tau^h_t \left( s^t h_t + (1 - s^t) h'_t \right) \right] + \beta^t \lambda_{t,7} b_0^t
\]

where \( \beta^* \) is either \( \beta \) or \( \beta' \). More precisely, the FOCs with respect to taxes and variables of the unconstrained household are derived under \( \beta^* = \beta \), e.g. \( \frac{\partial J}{\partial c_t} |_{\beta^* = \beta} \), and the ones with respect to variables of the unconstrained household under \( \beta^* = \beta' \), e.g. \( \frac{\partial J}{\partial c_t} |_{\beta^* = \beta'} \). The resulting FOCs differ from the previous ones only in that way
that here \( \beta = \overline{\beta} = 1 \) and \( \overline{\beta} \) is replaced by \( \beta \) or \( \beta' \), respectively, leading to

\[
\lambda_1 c^{-\mu c} + \lambda_3 c^{\mu c} + \lambda_5 h' + \lambda_7 h^T = 0
\]

\[
\lambda_2 + \lambda_3 - \lambda_5 n' - \lambda_7 n^T = 0
\]

\[
\frac{s'c}{\mu c} + \lambda_1 (\tau^h + \delta_h) + \lambda_2 (1 - \tau^n) c^{\mu c} + \lambda_4 m c^{-\mu c} c^{\mu c} (1 - \beta) + \lambda_5 m h' c^{\mu c} (\beta - 1) - \lambda_6 \frac{s'c^{\mu c+1}}{\mu c} - \lambda_7 \left[ g - \tau^h h^T - \tau^n n^T \right] = 0
\]

\[
h^{-\mu h} \left( 1 - \lambda_1 \frac{\mu h}{s'h} \right) + \lambda_6 [\beta (1 - \delta_h) - 1] - \lambda_7 \tau^h = 0
\]

\[
-n'^n + \lambda_2 \frac{\mu^n (1 - \tau^n)}{s'n} + \lambda_6 - \lambda_7 \tau^n = 0
\]

\[
\frac{(1 - s') c'}{\mu c} + \lambda_3 (1 - \tau^n) c^{\mu c} + \lambda_4 \left[ m (\beta - 1) + \tau^h + \delta_h \right] + \lambda_5 \frac{\mu c^{\mu c - 1}}{\mu c^{\mu c - 1}} - \lambda_6 (1 - s') = 0
\]

\[
h^{-\mu h} \left( 1 - \lambda_4 \frac{\mu h}{(1 - s') h'} \right) + \lambda_5 \left[ \frac{\beta' (1 - \delta_h) - 1 - \tau^h + m (\beta - \beta')}{(1 - s')} \right]
\]

\[
+ \lambda_6 [\beta' (1 - \delta_h) - 1] - \lambda_7 \tau^h = 0
\]

\[-n'^n + \lambda_3 \frac{\mu^n (1 - \tau^n)}{1 - s'}) n' + \lambda_5 \frac{(1 - \tau^n)}{(1 - s')} + \lambda_6 - \lambda_7 \tau^n = 0
\]

in the steady state.

### 3.5.1 Simple Tax System \((FP1)\)

For comparison, we expand the numerical results of table 2 with the ones of this alternative approach. Remind that the parameter values are given by \( \beta = 0.99 \), \( \mu^h = 3.7 \), \( \mu^c = 2 \), \( \mu^n = 0.2 \), \( \delta_h = 0.01 \), \( \beta' = 0.95 \), \( m = 0.7 \) and \( s' = 0.5 \).

As table 11 shows, this formulation leads to higher values of social welfare compared to the former formulation with an aggregate discount rate and delivers in the simple tax system a housing subsidy for \( g = b^g = 0 \). being more consistent with
Table 11: Numerical Results - Comparison.

<table>
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<tr>
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<th>Benchmark Model</th>
<th>Augm. Model &amp; FP1</th>
<th>Alt. Formulation &amp; FP1</th>
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</tr>
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</table>

the intuition. For $g = 0.45$ and $b^g = 0.8$ the resulting tax rates are very close to the ones of the benchmark model and differ from the baseline formulation results in that way that the housing tax is lower and the labor income tax is higher.

In order to sketch briefly the workings of this model and its advantages, I summarize the results in table 12 with $\beta = 0.99$, $\mu^h = 3.7$, $\mu^c = 2$, $\mu^n = 0.2$, $\delta_h = 0.01$ and $g = b^g = 0$.

The most important result in table 12 is that for $g = b^g = 0$, there is need for taxation if and only if private lending takes place ($m > 0$) and thus the collateral constraint becomes relevant. Thus, the considerations of discount factors and population shares become irrelevant as columns 2, 3 and 5 of table 12 show. In the absence of the confusing effects of discounting, the redistributinal purpose of optimal taxation becomes clearer. If private lending takes place, housing should be subsidized due to the collateral constraint and labor income taxed. As column 4 and 6 show, this subsidy and this tax should be the higher, the higher the collateralizable share of housing, $m$, is.
Table 12: Numerical Results - Alternative Formulation.

<table>
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</tr>
<tr>
<td>$n'$</td>
<td>1.0262</td>
<td>1.0195</td>
<td>1.0283</td>
<td>1.0195</td>
</tr>
<tr>
<td>$\tau^h$</td>
<td>0</td>
<td>0</td>
<td>-0.0030</td>
<td>0</td>
</tr>
<tr>
<td>$\tau^n$</td>
<td>0</td>
<td>0</td>
<td>0.0074</td>
<td>0</td>
</tr>
<tr>
<td>$u$</td>
<td>-1.8835</td>
<td>-1.8928</td>
<td>-1.8984</td>
<td>-1.8947</td>
</tr>
</tbody>
</table>

3.5.2 Elaborate Tax System (FP2)

Again we expand the numerical results of table 11 with the ones the alternative approach and the tax system FP2. Remind that the parameter values are given by $\beta = 0.99$, $\mu^b = 3.7$, $\mu^c = 2$, $\mu^n = 0.2$, $\delta_h = 0.01$, $\beta' = 0.95$, $m = 0.7$ and $s^l = 0.5$.

Table 13: Numerical Results - Comparison (FP2).

<table>
<thead>
<tr>
<th>Benchmark Model</th>
<th>Augm. Model &amp; FP2</th>
<th>Alt. Formulation &amp; FP2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g = 0$</td>
<td>$g = 0.45$</td>
<td>$g = 0$</td>
</tr>
<tr>
<td>$b^g = 0$</td>
<td>$b^g = 0.8$</td>
<td>$b^g = 0$</td>
</tr>
<tr>
<td>$c$</td>
<td>0.9974</td>
<td>0.7907</td>
</tr>
<tr>
<td>$h$</td>
<td>2.8785</td>
<td>2.3725</td>
</tr>
<tr>
<td>$n$</td>
<td>1.0262</td>
<td>1.0625</td>
</tr>
<tr>
<td>$c'$</td>
<td>-</td>
<td>0.9911</td>
</tr>
<tr>
<td>$h'$</td>
<td>-</td>
<td>2.3100</td>
</tr>
<tr>
<td>$n'$</td>
<td>-</td>
<td>0.9560</td>
</tr>
<tr>
<td>$\tau^h(l,j)$</td>
<td>0</td>
<td>0.0130</td>
</tr>
<tr>
<td>$\tau^n$</td>
<td>0</td>
<td>0.3388</td>
</tr>
<tr>
<td>$\tau^h,n$</td>
<td>-</td>
<td>-0.0432</td>
</tr>
<tr>
<td>$u$</td>
<td>-1.8835</td>
<td>-2.4085</td>
</tr>
</tbody>
</table>

Also in the tax system $FP2$ the alternative formulation leads to higher values of social welfare. Optimal taxation exhibits a housing subsidy for the borrower and a labor income tax for $g = b^g = 0$, while the lenders’ housing tax is slightly
Table 14: Numerical Results with FP2 - Alternative Formulation.

<table>
<thead>
<tr>
<th></th>
<th>$\beta' = 0.98$</th>
<th></th>
<th>$\beta' = 0.95$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$m = 0$, $s^l = 0.5$</td>
<td>$m = 0$, $s^l = 0.5$</td>
<td>$m = 0$, $s^l = 0.7$</td>
</tr>
<tr>
<td>$c$</td>
<td>0.9974</td>
<td>0.9975</td>
<td>0.9843</td>
</tr>
<tr>
<td>$h$</td>
<td>2.8792</td>
<td>2.8804</td>
<td>2.8507</td>
</tr>
<tr>
<td>$n$</td>
<td>1.0262</td>
<td>1.0261</td>
<td>1.0255</td>
</tr>
<tr>
<td>$c'$</td>
<td>0.9977</td>
<td>0.9981</td>
<td>0.9873</td>
</tr>
<tr>
<td>$h'$</td>
<td>2.5806</td>
<td>2.1404</td>
<td>2.1047</td>
</tr>
<tr>
<td>$n'$</td>
<td>1.0235</td>
<td>1.0197</td>
<td>0.9956</td>
</tr>
<tr>
<td>$\tau_{h, b}$</td>
<td>0</td>
<td>0</td>
<td>0.0002</td>
</tr>
<tr>
<td>$\tau_{h, b}$</td>
<td>0</td>
<td>0</td>
<td>0.0282</td>
</tr>
<tr>
<td>$\tau_{h, b}$</td>
<td>0</td>
<td>0</td>
<td>-0.0254</td>
</tr>
<tr>
<td>$u$</td>
<td>-1.8857</td>
<td>-1.8929</td>
<td>-1.8941</td>
</tr>
</tbody>
</table>

positive. For $g = 0.45$ and $b^g = 0.8$ the resulting optimal housing tax for the lender is smaller and the one for the borrower larger than in the former formulation. Here, another interesting point appears, namely that the borrower’s housing tax rate is larger than the lender’s tax rate. In this formulation, there is a threshold for which the borrowers tax gets larger than the lenders one, as we will see below.

I summarize further results for the alternative formulation with $FP2$ in table 14 with $\beta = 0.99$, $\mu^h = 3.7$, $\mu^c = 2$, $\mu^n = 0.2$, $\delta_h = 0.01$ and $g = b^g = 0$.

Here we see, that for $g = b^g = 0$ there is no need for taxation in the absence of private borrowing, if the shares are equal. For non-identic shares the tax rates differ from zero. The optimal borrower housing tax becomes negative and the lender tax positive. For $m > 0$ the government subsidizes the borrower since they face a binding borrowing constraint. In the last two columns we see that with a rise in government spending all tax rates rise in absence of private borrowing. The tax rate for the borrower rises even faster than the on for the lender. Since the borrower
is more inelastic with a lower level of housing demand, the government taxes him stronger to finance his given expenditures. For the last column in table 13 this last effect is stronger than the former one leading to $\tau^{h,b} > \tau^{h,l}$.

4 Conclusion

The paper provides a theoretical framework to investigate optimal fiscal policy with respect to housing. Although research on real estate economics grew rapidly since the recent crisis, the literature on optimal fiscal policy typically ignores real estate that plays a dual role for fiscal policy being taxed on the one hand as well as subsidized on the other hand.

In a two-sector general equilibrium model with non-durable consumption goods and durable housing, the paper shows how optimal tax rates have to be set in order to maximize a utilitarian social welfare function.

The first result in a representative agent framework is that – corresponding to the principles of optimal capital income taxation – a housing tax should be small in modulus in the long run compared to a labor income tax due to the intertemporal distortions associated with taxation of durable housing. Depending on the utility function it can be positive, negative or exactly zero, but is always small in modulus.

Another important finding of the paper is that in a framework with capital market imperfections, where collateral constrained borrowers use real estate as collateral optimal fiscal policy differs from the benchmark results. This is another parallelity to the capital income tax literature, where the famous zero capital income tax result does no more hold in the presence of liquidity constraints. In the augmented model
with imperfect capital markets optimal fiscal policy should disburden constrained agents by subsidizing their housing, while taxing housing of the unconstrained ones and labor income.
References


5 Appendix

5.1 Derivation of the Implementability Constraint

In order to get the implementability constraint we solve (5) for \( R^g_t \)

\[
R^g_t = \frac{(1 - \delta_h) p_{h,t+1}}{(1 + \tau^h_t) p_{h,t} - \frac{v^h_t}{u^h_t}}
\]  

(14)

and rewrite condition (4):

\[
u^c_t = u^c_{t-1} (\beta R^g_{t-1})^{-1} \quad \text{and} \quad u^c_{t-1} = u^c_{t-2} (\beta R^g_{t-2})^{-1} \Rightarrow u^c_t = u^c_{t-2} \beta^{-2} (R^g_{t-1})^{-1} (R^g_{t-2})^{-1}.
\]

Iterating forward we get

\[
\beta^n u^c_t = u^c_0 \prod_{i=0}^{t-1} (R^g_i)^{-1}.
\]  

(15)

Thus we can rewrite the transversality condition as (with \( u^c_0 > 0 \))

\[
\lim_{t \to \infty} E_t \prod_{i=0}^{t-1} (R^g_i)^{-1} \frac{b^g_{t+1}}{R^g_t} = 0.
\]

Now we solve the household budget constraint for period \( t + 1 \) for \( b^g_{t+1} \)

\[
b^g_{t+1} = c_{t+1} + (1 + \tau^h_{t+1}) p_{h,t+1} h_{t+1} + \frac{b^g_{t+2}}{R^g_{t+1}} - (1 - \tau^n_{t+1}) w_{t+1} n_{t+1} - (1 - \delta_h) p_{h,t+1} h_t
\]

and insert this in the one for period \( t \) and get

\[
c_t + (1 + \tau^h_t) p_{h,t} h_t + \frac{1}{R^g_t} \left[ c_{t+1} + (1 + \tau^h_{t+1}) p_{h,t+1} h_{t+1} + \frac{b^g_{t+2}}{R^g_{t+1}} \right] - b^g_t
\]

\[
= (1 - \tau^n_t) w_t n_t + (1 - \delta_h) p_{h,t} h_{t-1}.
\]
or rewritten
\[ c_t + \frac{c_{t+1}}{R_t^g} + (1 + \tau_t^h) p_{h,t} h_t - \frac{(1 - \delta_t) p_{h,t+1} h_t}{R_t^g} + \frac{(1 + \tau_{t+1}^h) p_{h,t+1} h_{t+1}}{R_t^g} + \frac{b_{t+2}^g}{R_t^g R_{t+1}^g} = (1 - \tau_t^n) w_t n_t + \frac{(1 - \tau_{t+1}^n) w_{t+1} n_{t+1}}{R_t^g} + (1 - \delta_t) p_{h,t} h_{t-1} + b_t^g. \]

We now take a look at the terms with \( h_t \) and insert (14)
\[
\begin{align*}
  h_t \left[ (1 + \tau_t^h) p_{h,t} - \frac{(1 - \delta_t) p_{h,t+1}}{R_t^g} \right] \\
  = (14) h_t \left[ (1 + \tau_t^h) p_{h,t} - \frac{(1 - \delta_t) p_{h,t+1}}{(1 - \delta_t) p_{h,t+1}} \left( (1 + \tau_t^h) p_{h,t} - \frac{w_t^h}{u_t^h} \right) \right] \\
  = h_t \left[ (1 + \tau_t^h) p_{h,t} - \left( (1 + \tau_t^h) p_{h,t} - \frac{w_t^h}{u_t^h} \right) \right] = h_t \frac{w_t^h}{u_t^h}.
\end{align*}
\]

Thus we can rewrite the budget constraint again and get
\[
\begin{align*}
  c_t + \frac{c_{t+1}}{R_t^g} + h_t \frac{w_t^h}{u_t^h} + \frac{(1 + \tau_{t+1}^h) p_{h,t+1} h_{t+1}}{R_t^g} + \frac{b_{t+2}^g}{R_t^g R_{t+1}^g} \\
  = (1 - \tau_t^n) w_t n_t + \frac{(1 - \tau_{t+1}^n) w_{t+1} n_{t+1}}{R_t^g} + (1 - \delta_t) p_{h,t} h_{t-1} + b_t^g.
\end{align*}
\]

Inserting the budget constraint of \( t + 2 \) then delivers
\[
\begin{align*}
  c_t + \frac{c_{t+1}}{R_t^g} + h_t \frac{w_t^h}{u_t^h} + \frac{(1 + \tau_{t+1}^h) p_{h,t+1} h_{t+1}}{R_t^g} \\
  + \frac{1}{R_t^g R_{t+1}^g} \left[ c_{t+2} + (1 + \tau_{t+2}^h) p_{h,t+2} h_{t+2} + \frac{b_{t+3}^g}{R_{t+2}^g} - (1 - \tau_{t+2}^n) w_{t+2} n_{t+2} - (1 - \delta_t) p_{h,t+2} h_{t+1} \right] \\
  = (1 - \tau_t^n) w_t n_t + \frac{(1 - \tau_{t+1}^n) w_{t+1} n_{t+1}}{R_t^g} + (1 - \delta_t) p_{h,t} h_{t-1} + b_t^g.
\end{align*}
\]
and rewritten

\[ c_t + \frac{c_{t+1}}{R_t^g} + \frac{c_{t+2}}{R_t^g R_{t+1}^g} + h_t \frac{u_t^h}{u_t^c} + \left( 1 + \tau_{t+1}^h \right) \frac{p_{h,t+1} h_{t+1}}{R_t^g} - \left( 1 - \delta_h \right) p_{h,t+2} h_{t+1} \]

\[ + \frac{1 + \tau_{t+2}^h}{R_t^g R_{t+1}^g} p_{h,t+2} h_{t+2} \]

\[ + \frac{b_{t+3}^g}{R_t^g R_{t+1}^g R_{t+2}^g} = \left( 1 - \tau_t^c \right) w_t n_t + \frac{1 - \tau_{t+1}^n}{R_t^g} w_{t+1} n_{t+1} \]

\[ + \frac{1 - \tau_{t+2}^n}{R_t^g R_{t+1}^g R_{t+2}^g} w_{t+2} n_{t+2} + \left( 1 - \delta_h \right) p_{h,t} h_{t-1} + b_t^g. \]

Doing the steps above in (16) we get

\[ c_t + \frac{c_{t+1}}{R_t^g} + \frac{c_{t+2}}{R_t^g R_{t+1}^g} + h_t \frac{u_t^h}{u_t^c} + \frac{h_{t+1} u_{t+1}^h}{R_t^g} + \frac{1 + \tau_{t+2}^h}{R_t^g R_{t+1}^g} p_{h,t+2} h_{t+2} + \frac{b_{t+3}^g}{R_t^g R_{t+1}^g R_{t+2}^g} \]

\[ = \left( 1 - \tau_t^c \right) w_t n_t + \frac{1 - \tau_{t+1}^n}{R_t^g} w_{t+1} n_{t+1} + \frac{1 - \tau_{t+2}^n}{R_t^g R_{t+1}^g} w_{t+2} n_{t+2} + \left( 1 - \delta_h \right) p_{h,t} h_{t-1} + b_t^g. \]

By doing this successively we get the intertemporal budget constraint, where bond holdings disappear due to the transversality condition with the initial endowment of \( h_{-1} \) and \( b_0^g \)

\[ \sum_{t=0}^{\infty} \left( \prod_{i=0}^{t-1} \left( R_i^g \right)^{-1} \right) c_t + \sum_{t=0}^{\infty} \left( \prod_{i=0}^{t-1} \left( R_i^g \right)^{-1} \right) h_t \frac{u_t^h}{u_t^c} \]

\[ = \sum_{t=0}^{\infty} \left( \prod_{i=0}^{t-1} \left( R_i^g \right)^{-1} \right) \left( 1 - \tau_t^c \right) w_t n_t + \left( 1 - \delta_h \right) p_{h,0} h_{-1} + b_0^g \]

or rewritten

\[ \sum_{t=0}^{\infty} \left( \prod_{i=0}^{t-1} \left( R_i^g \right)^{-1} \right) \left[ c_t + h_t \frac{u_t^h}{u_t^c} - \left( 1 - \tau_t^c \right) w_t n_t \right] + \left( 1 - \delta_h \right) p_{h,0} h_{-1} + b_0^g = 0. \]

By eliminating prices with (15) and (3) we get the implementability constraint

\[ \sum_{t=0}^{\infty} \beta^t u_0^t \left[ c_t + h_t \frac{u_t^h}{u_t^c} + \frac{u_t^p}{u_t^c} n_t \right] + \left( 1 - \delta_h \right) p_{h,0} h_{-1} + b_0^g = 0. \]

\[ \Leftrightarrow \sum_{t=0}^{\infty} \beta^t \left[ u_t^c c_t + u_t^h h_t + u_t^p n_t \right] + \left( 1 - \delta_h \right) p_{h,0} h_{-1} + b_0^g = 0. \]  

(17)
5.2 Ramsey-Problem: Housing Tax

The FOC of the Ramsey problem wrt housing is given by

\[
\frac{\partial J}{\partial h_t} = 0 \Rightarrow h_t^{-\mu_h} + \phi (1 - \mu^h) h_t^{-\mu_h} - \rho_t + \beta E_t \rho_{t+1} (1 - \delta_h) = 0
\]

\[
\Leftrightarrow \rho_t = h_t^{-\mu_h} + \phi (1 - \mu^h) h_t^{-\mu_h} + E_t \rho_{t+1} \beta (1 - \delta_h)
\]

\[
\Rightarrow (7) c_t^{-\mu^c} [1 + \phi (1 - \mu^c)] = h_t^{-\mu_h} + \phi (1 - \mu^h) h_t^{-\mu_h}
\]

\[
+ E_t c_{t+1}^{-\mu^c} [1 + \phi (1 - \mu^c)] \beta (1 - \delta_h)
\]

\[
\Rightarrow [1 + \phi (1 - \mu^c)] = \frac{h_t^{-\mu_h} + E_t c_{t+1}^{-\mu^c} \beta (1 - \delta_h)}{c_t^{-\mu^c} (1 + \tau_t^h)}
\]

\[
+ \phi (1 - \mu^h) \frac{h_t^{-\mu_h}}{c_t^{-\mu^c}} + \frac{E_t c_{t+1}^{-\mu^c} \beta}{c_t^{-\mu^c} \phi (1 - \mu^c) (1 - \delta_h)} \frac{1}{1/R_t^c}
\]
The optimal tax rate on housing follows from (10)

\[ \tau_t^h = \phi (1 - \mu^c) - \phi (1 - \mu^h) \frac{h_t^{-\mu^h}}{c_t^{-\mu^c}} - \phi (1 - \mu^c) (1 - \delta_h) \frac{1}{R_t^g} \]

\[ = \phi \left[ 1 - \mu^c - (1 - \mu^h) \frac{h_t^{-\mu^h}}{c_t^{-\mu^c}} - \frac{(1 - \mu^c) (1 - \delta_h)}{R_t^g} \right] \]

\[ = \phi \left[ 1 - \mu^c - (1 - \mu^h) \left(1 + \tau_t^h\right) - \frac{(1 - \mu^h) (1 - \delta_h)}{R_t^g} \right] - \frac{(1 - \mu^c) (1 - \delta_h)}{R_t^g} \]

\[ = \phi \left[ 1 - \mu^c - (1 - \mu^h) \left(1 + \tau_t^h\right) + \frac{(1 - \mu^h) (1 - \delta_h)}{R_t^g} - \frac{(1 - \mu^c) (1 - \delta_h)}{R_t^g} \right] \]

\[ = \phi \left[ 1 - \mu^c - 1 + \mu^h - (1 - \mu^h) \tau_t^h + \frac{(1 - \mu^h - 1 + \mu^c) (1 - \delta_h)}{R_t^g} \right] \]

\[ \Rightarrow \tau_t^h \left[ 1 + \phi (1 - \mu^h) \right] = \phi \left[ \mu^h - \mu^c - \frac{(\mu^h - \mu^c) (1 - \delta_h)}{R_t^g} \right] \]

\[ \tau_t^h = \frac{\phi}{1 + \phi (1 - \mu^h)} \left( \mu^h - \mu^c \right) \left( 1 - \frac{(1 - \delta_h)}{R_t^g} \right), \quad >0, \text{ small} \]

with \( \frac{h_t^{-\mu^h}}{c_t^{-\mu^c}} = (1 + \tau_t^h) - \frac{(1 - \delta_h)}{R_t^g} \) following from (14).

5.2.1 Second Derivatives

The second derivatives wrt housing are given by

\[ \frac{\partial^2 J}{\partial h_t^2} = -\mu^h h_t^{-1 - \mu^h} \left[ 1 - \phi (\mu^h - 1) \right]. \]

Whether this expression is positive or negative depends on the last term. We get

\[ \frac{\partial^2 J}{\partial h_t^2} < 0 \] and thus a maximum for

\[ 1 - \phi (\mu^h - 1) > 0 \Leftrightarrow \phi < \frac{1}{(\mu^h - 1)}. \]
For $c$ and $n$ the second derivatives are always positive for $1 \leq \mu^c \leq 2$, which is the case in the benchmark calibration,

$$\frac{\partial^2 J}{\partial c_i^2} = -\left\{\frac{c_i^{1-\mu^c}}{1 - \phi (\mu^c - 1)}\right\} < 0$$

$$\Leftrightarrow \phi < \frac{1}{(\mu^c - 1)} \geq 1 \text{ for } 1 \leq \mu^c \leq 2,$$

and $$\frac{\partial^2 J}{\partial n_i^2} = -\left\{\frac{n_i^{1+\mu^n}}{1 + \phi (\mu^n + 1)}\right\} < 0.$$

### 5.2.2 Shape of the Tax Rates

The labor income tax in conave in $\phi$ for $\mu^c \geq 1$ since

$$\frac{\partial^2 \tau^n}{\partial \phi} = \frac{1 + \phi + \phi \mu^n - 2 \mu^n - 2 \mu^c - 2 (\mu^n)^2 - 2 \mu^c \mu^n}{[1 + \phi (1 + \mu^n)]^3}$$

$$1 + \phi - 2 \mu^c + \phi \mu^n - 2 \mu^n - 2 (\mu^n)^2 - 2 \mu^c \mu^n$$

$$= \frac{<0 \text{ for } \mu^c \geq 1}{[1 + \phi (1 + \mu^n)]^3} < 0.$$

The housing tax is convex for $\mu^c < \mu^h$ and $\mu^h > 1$ due to

$$\frac{\partial^2 \tau^h}{\partial \phi^2} = \frac{(\mu^h - \mu^c)}{[1 - \phi (\mu^h - 1)]^2} [1 - \beta (1 - \delta_h)]$$

$$1 - \phi (\mu^h - 1) + 2 (\mu^h - 1) (\mu^h - \mu^c) [1 - \beta (1 - \delta_h)]$$

$$\frac{>0 \text{ for } \phi < \frac{1}{(\mu^h - 1)}}{[1 - \phi (\mu^h - 1)]^3} > 0.$$
5.3 Summary of Conditions

Summarizing the constraints faced by the government, while choosing optimal tax
rates, delivers

\[ h_t^{-\mu^h} = (1 + \tau_t^h) p_{h,t} c_t^{-\mu^e} - E_t \beta c_{t+1}^{-\mu^e} (1 - \delta_h) p_{h,t+1} \]

\[ n_t^{-\mu^n} = (1 - \tau_t^n) w_t c_t^{-\mu^e} \]

\[ c_t^{-\mu^e} = E_t \beta R_t c_{t+1}^{-\mu^e} \]

\[ c_t^{-\mu^e} = E_t \beta R_t c_{t+1}^{-\mu^e} \]

\[ h_t^{-\mu^h} = (1 + \tau_t^h) p_{h,t} c_t^{-\mu^e} - E_t \beta c_{t+1}^{-\mu^e} (1 - \delta_h) p_{h,t+1} + \omega_t m E_t p_{h,t+1} \]

\[ n_t^{-\mu^n} = (1 - \tau_t^n) w_t c_t^{-\mu^e} \]

\[ \omega_t = \frac{c_t^{-\mu^e} - \beta' E_t c_{t+1}^{-\mu^e} R_t}{R_t} \]

\[ c_t' + (1 + \tau_t^h) p_{h,t} h_t' = (1 - \tau_t^n) w_t n_t' + (1 - \delta_h) p_{h,t} h_{t-1}' - \frac{b_{t+1}'}{R_t} + b_t' \]

\[ 0 = \sum_{t=0}^{\infty} \left( \prod_{i=0}^{t-1} (R_i)^{-1} \right) [g_t - \tau_t^h p_{h,t} h_t - \tau_t^n w_t n_t] + b_0^g \]

\[ y_{c,t} = n_{c,t}^T, \quad y_{h,t} = n_{h,t}^T, \quad w_t = 1, \quad p_{h,t} = 1 \]

\[ h_t^T = s'h_t + (1 - s') h_t', \quad n_t^T = s'n_t + (1 - s') n_t', \quad c_t^T = s' c_t + (1 - s') c_t' \]

\[ n_t^T = n_{c,t}^T + n_{h,t}^T, \quad b_{t+1} \geq -m E_t [p_{h,t+1} h_t'] \]

\[ c_t^T + g_t + p_{h,t} h_t^T = y_{c,t} + p_{h,t} y_{h,t} + (1 - \delta_h) p_{h,t} h_{t-1}^T \]

Using this relationship following from the euler equation for bonds

\[ c_t^{-\mu^e} = c_{t-1}^{-\mu^e} (\beta R_{t-1}^g)^{-1} \]

and

\[ c_{t-1}^{-\mu^e} = c_{t-2}^{-\mu^e} (\beta R_{t-2}^g)^{-1} \]

\[ \Rightarrow c_t^{-\mu^e} = c_{t-2}^{-\mu^e} \beta^{-2} (R_{t-1}^g)^{-1} (R_{t-2}^g)^{-1} \]

\[ \Rightarrow \frac{c_t^{-\mu^e}}{c_0^{-\mu^e}} = \prod_{i=0}^{t-1} (R_i^g)^{-1} \]

54
the conditions above can be reduced to

\[ h_t^{-\mu h} = (1 + \tau_t^h) c_t^{-\mu c} - \beta (1 - \delta_h) E_t c_{t+1}^{-\mu c} \]

\[ n_t^{\mu n} c_t^{\mu c} = (1 - \tau_t^n) \]

\[ R_t^g = R_t = \frac{c_t^{-\mu c}}{\beta E_t c_{t+1}^{-\mu c}} \]

\[ h_t^{-\mu h} = (1 + \tau_t^h) c_t^{t-\mu c} - \beta' (1 - \delta_h) E_t c_{t+1}^{t-\mu c} + m \left[ \frac{c_t^{t-\mu c}}{c_t^{-\mu c}} \beta E_t c_{t+1}^{-\mu c} - \beta' E_t c_{t+1}^{t-\mu c} \right] \]

\[ n_t^{\mu n} c_t^{\mu c} = (1 - \tau_t^n) \]

\[ c_t^{t+1} + (1 + \tau_t^h) h_t^{t+1} = (1 - \tau_t^n) n_t^{t+1} + (1 - \delta_h) h_{t-1}^{t+1} + \frac{m h_t^0}{c_t^{t-\mu c}} \beta E_t c_{t+1}^{-\mu c} - m h_{t-1}^0 \]

\[ 0 = \sum_{t=0}^{\infty} \left( \frac{\beta'}{c_0^{-\mu c}} \right) \left[ g_t - \tau_t^h \left( s^t h_t + (1 - s^t) h_t^{t+1} \right) - \tau_t^n \left( s^t n_t + (1 - s^t) n_t^{t+1} \right) \right] + b_0^g \]

\[ s^t c_t^{t+1} + (1 - s^t) c_t^{t+1} + g_t + s^t h_t + (1 - s^t) h_t^{t+1} = s^t n_t + (1 - s^t) n_t^{t+1} + (1 - \delta_h) \left( s^t h_{t-1} + (1 - s^t) h_{t-1}^{t+1} \right) \]

given \( b_0^g \) and \( b_0 \).

### 5.4 Solution of the Ramsey Problem and the Steady State

The FOCs of the Ramsey problem are given by

\[ \frac{\partial J}{\partial \tau_t^c} = 0 \Rightarrow -\beta \lambda_{t,1} c_t^{-\mu c} - \beta' \lambda_{t,4} c_t^{t-\mu c} - \beta' \lambda_{t,5} h_t' - \beta \lambda_t c_t^{-\mu c} h_T^T = 0 \]

\[ \Rightarrow \left( -\beta' \right) \lambda_{t,1} c_t^T - \lambda_{t,4} c_t^{T-t} - \lambda_{t,5} h_T' + \beta' \lambda_t c_t^{-\mu c} h_T^T = 0, \]

with \( \beta' = \left( \frac{\beta}{\beta'} \right)^t = \left( \frac{\beta}{\beta' \beta (1-s^t)} \right)^t = \left[ \left( \frac{\beta}{\beta'} \right)^{(1-s^t)} \right]^t. \)

\[ \frac{\partial J}{\partial \tau_t^t} = 0 \Rightarrow \lambda_{t,2} + \lambda_{t,3} - \lambda_{t,5} n_t' - \beta \lambda_t c_t^{-\mu c} h_T^T = 0 \]

55
\[
\frac{\partial J}{\partial c_t} = 0 \Rightarrow \beta^t s^t c_t^{-\mu^e} + \beta^t \lambda_{t,1} (1 + \tau_t^h) \mu^e c_t^{-\mu^e-1} + \beta^t \lambda_{t,2} n_t^\mu^e c_t^{-\mu^e-1} - \beta^t \lambda_{t,4} m c_t^{-\mu^e-1} \beta E_t c_{t+1}^{-\mu^e} c_t^{-\mu^e-1} \\
+ \beta^t \lambda_{t,5} m h_t^\mu^e c_t^{-\mu^e-1} - \beta^t \lambda_{t,6} s^t - \beta^t \lambda_7 \frac{c_t^{-\mu^e-1}}{c_0^{-\mu^e}} \mu^e \left[ g_t - \tau_t^h h_t^r - \tau_{t,n_t}^r \right] \\
- \beta^t \lambda_{t,1,1} (1 - \delta_h) \mu^e c_t^{-\mu^e-1} + \beta^t \lambda_{t,4} m c_{t-1}^{-\mu^e} c_t^{-\mu^e-1} \\
- \beta^t \lambda_{t,1,5} \frac{m h_t^\mu e c_t^{-\mu^e-1}}{c_t^{-\mu^e-1}} = 0 \\
\Rightarrow \beta^t \mu^e c_t^{-\mu^e-1} > 0 \beta^t s^t c_t^{-\mu^e} + \lambda_{t,1} (1 + \tau_t^h) + \lambda_{t,2} n_t^\mu^e c_t^{-\mu^e} - \lambda_{t,4} m c_t^{-\mu^e} \beta E_t c_{t+1}^{-\mu^e} \\
+ \lambda_{t,5} m h_t^\mu^e c_t^{-\mu^e} - \lambda_{t,6} \frac{s^t c_t^{-\mu^e}}{c_t^{-\mu^e-1}} - \beta^t \lambda_7 \frac{c_t^{-\mu^e-1}}{c_0^{-\mu^e}} \mu^e \left[ g_t - \tau_t^h h_t^r - \tau_{t,n_t}^r \right] \\
- \beta^t \lambda_{t,1,1} (1 - \delta_h) + \beta^t \lambda_{t,4} m c_{t-1}^{-\mu^e} c_t^{-\mu^e-1} - \beta^t \lambda_{t,1,5} m h_t^\mu e c_t^{-\mu^e-1} = 0 \\
\frac{\partial J}{\partial h_t} = 0 \Rightarrow \beta^t s^t h_t^{-\mu^h} - \beta^t \lambda_{t,1} h_t^{-\mu^b} c_t^{-\mu^h-1} - \beta^t \lambda_{t,6} s^t - \beta^t \lambda_7 \frac{c_t^{-\mu^e}}{c_0^{-\mu^e}} \tau_t^h + \lambda_{t,1,6} \tilde{\beta} \lambda_{t,1,6} (1 - \delta_h) s^t = 0 \\
\Rightarrow \beta^t s^t > 0 \beta^t \frac{h_t^{-\mu^h}}{s^t} - \lambda_{t,1} \frac{h_t^{-\mu^h-1}}{s^t} - \lambda_{t,6} - \beta^t \lambda_7 \frac{c_t^{-\mu^e}}{c_0^{-\mu^e}} \tau_t^h + \lambda_{t,1,6} (1 - \delta_h) = 0 \\
\frac{\partial J}{\partial n_t} = 0 \Rightarrow -\beta^t s^t n_t^\mu^e + \beta^t \lambda_{t,2} c_t^{-\mu^e} n_t^\mu - \beta^t \lambda_{t,6} s^t - \beta^t \lambda_7 \frac{c_t^{-\mu^e}}{c_0^{-\mu^e}} \tau_t^h = 0 \\
\Rightarrow \beta^t s^t > 0 - \beta^t n_t^\mu^e + \lambda_{t,2} \frac{n_t^\mu}{s^t} - \beta^t \lambda_{t,6} s^t - \beta^t \lambda_7 \frac{c_t^{-\mu^e}}{c_0^{-\mu^e}} \tau_t^h = 0 \\
\frac{\partial J}{\partial c_t} = 0 \Rightarrow \beta^t n_t (1 - s^t) c_t^{-\mu^e} + \beta^t \lambda_{t,3} n_t^\mu^e c_t^{-\mu^e-1} + \beta^t \lambda_{t,4} (1 + \tau_t^h) c_t^{-\mu^e-1} - \beta^t \lambda_{t,4} m c_t^{-\mu^e-1} c_t^{-\mu^e} \beta E_t c_{t+1}^{-\mu^e} \\
- \beta^t \lambda_{t,5} - \beta^t \lambda_{t,6} (1 - s^t) - \beta^t \lambda_{t,1,4} \beta^t (1 - \delta_h) c_t^{-\mu^e-1} - \beta^t \lambda_{t,1,4} m c_t^{-\mu^e-1} c_t^{-\mu^e} = 0 \\
\Rightarrow \beta^t \mu^e c_t^{-\mu^e-1} > 0 \beta^t \frac{(1 - s^t) c_t^e}{c_t^e} + \lambda_{t,3} n_t^\mu^e c_t^{2\mu^e} + \lambda_{t,4} \left[ (1 + \tau_t^h) + m c_t^{\mu^e} \beta E_t c_{t+1}^{-\mu^e} \right] - \frac{\lambda_{t,5}}{c_t^e} \frac{1}{c_t^e} - \beta \lambda_{t,1,4} \left[ 1 - \delta_h + m \right] = 0,
with \( \vec{\beta}' = \frac{\vec{\beta}}{\beta'} = \left( \frac{\vec{\beta}}{\beta'} \right)^t = \left( \frac{\vec{\beta}}{\beta'} \right)^s \).

\[
\frac{\partial J}{\partial h'_t} = 0 \Rightarrow \beta^n (1 - s') h'^{t-\mu}_t - \vec{\beta}^{t} \lambda_{t,4} h'^{t-\mu}_t - \vec{\beta}^{t} \lambda_{t,5} (1 + \tau'^{h}_t) + \vec{\beta}^{t} \lambda_{t,6} (1 - s') = 0
\]

\[
\Rightarrow \vec{\beta}^{t} (1 - s') h'^{t-\mu}_t - \lambda_{t,5} \frac{\mu'^{h}}{(1 - s')} h'^{t-\mu}_t - \frac{\lambda_{t,5}}{(1 - s')} [(1 + \tau'^{h}_t) - m c'^{t \mu}_{t} E_t c^{-t \mu}_{t+1}] - \lambda_{t,6} = 0
\]

\[
\frac{\partial J}{\partial n'_t} = 0 \Rightarrow -\beta^n (1 - s') n'^{t\mu}_t + \vec{\beta}^{t} \lambda_{t,3} c^{t \mu}_{t} n'^{t\mu}_t - \vec{\beta}^{t} \lambda_{t,5} (1 - n'^{t\mu}_t) + \vec{\beta}^{t} \lambda_{t,6} (1 - s') = 0
\]

\[
\Rightarrow \vec{\beta}^{t} (1 - s') h'^{t-\mu}_t + \lambda_{t,5} \frac{\mu'^{h}_t}{(1 - s')} h'^{t-\mu}_t + \frac{\lambda_{t,5}}{(1 - s')} [(1 + \tau'^{h}_t) - m c'^{t \mu}_{t} E_t c^{-t \mu}_{t+1}] - \lambda_{t,6} = 0.
\]
The FOCs of the Ramsey Problem can summarized by:

\[
\lambda_{t,1} c_{t}^{-\mu^e} + \lambda_{t,4} c_{t}^{-\mu^e} + \lambda_{t,5} h_{t}^l + \tilde{\beta}^t \lambda_7 \frac{c_{t}^{-\mu^e}}{c_0^{-\mu^e}} h_{t}^T = 0
\]

\[
\lambda_{t,2} + \lambda_{t,3} - \lambda_{t,5} n_{t}^l - \tilde{\beta}^t \lambda_7 \frac{c_{t}^{-\mu^e}}{c_0^{-\mu^e}} n_{t}^T = 0
\]

\[
\tilde{\beta}^t \frac{s^t c_t}{\mu^e} + \lambda_{t,1} \left(1 + \tau_t^h\right) + \lambda_{t,2} n_t^{\mu^u} \frac{c_t}{\mu^e} - \lambda_{t,4} m c_t^{-\mu^e} \beta E_t c_{t+1}^{-\mu^e} + \lambda_{t,5} m h_t^l \beta E_t c_{t+1}^{-\mu^e} - \lambda_{t,6} \frac{s^t c_t^{\mu^e+1}}{\mu^e} - \frac{\tilde{\beta}^t \lambda_7}{c_0^{-\mu^e}} \left[ g_t - \tau_t^h h_t^T - \tau_t^h n_t^T \right] - \lambda_{t,1} (1 - \delta_h) + \lambda_{t,4} m c_t^{-\mu^e} c_t^{-\mu^e} - \lambda_{t,5} m h_t^{-\mu^e} c_t^{-\mu^e} = 0
\]

\[
\tilde{\beta}^t h_t^{-\mu^h} + \lambda_{t,1} \frac{\mu^h}{s^t} h_t^{-\mu^h} - \lambda_{t,6} - \tilde{\beta}^t \lambda_7 \frac{c_t^{-\mu^e}}{c_0^{-\mu^e}} \tau_t^t + \lambda_{t,1} \beta (1 - \delta_h) = 0
\]

\[
-\tilde{\beta}^t n_t^{\mu^u} + \lambda_{t,2} \frac{\mu^u}{s^t} c_t^{\mu^e} n_t^{\mu^u-1} + \lambda_{t,6} - \tilde{\beta}^t \lambda_7 \frac{c_t^{-\mu^e}}{c_0^{-\mu^e}} \tau_t^t = 0
\]

\[
-\tilde{\beta}^t \left(1 - s^t \right) c_t^l + \lambda_{t,3} n_t^{\mu^u} c_t^{\mu^e} + \lambda_{t,4} \left[\left(1 + \tau_t^h\right) + m c_t^{\mu^e} \beta E_t c_{t+1}^{-\mu^e}\right] - \frac{\lambda_{t,5}}{\mu^e c_t^{\mu^e-1}} - \lambda_{t,6} \left(1 - s^t \right) - \beta \lambda_{t,1} \beta (1 - \delta_h + m) = 0
\]

\[
-\tilde{\beta}^t h_t^{-\mu^h} - \lambda_{t,4} \frac{\mu^h}{s^t} h_t^{-\mu^h} - \lambda_{t,5} \left[\left(1 + \tau_t^h\right) - m c_t^{\mu^e} \beta E_t c_{t+1}^{-\mu^e}\right] - \lambda_{t,6} - \lambda_{t,5} \left(1 - \delta_h + m\right) \frac{1}{s^t} + \lambda_{t,6} \lambda_{t,1} \beta (1 - \delta_h) = 0
\]

\[
-\tilde{\beta}^t n_t^{\mu^u} + \lambda_{t,3} \frac{\mu^u}{s^t} c_t^{\mu^e} n_t^{\mu^u-1} + \lambda_{t,5} \left(1 - \tau_t^u\right) \frac{1}{s^t} + \lambda_{t,6} - \tilde{\beta}^t \lambda_7 \frac{c_t^{-\mu^e}}{c_0^{-\mu^e}} \tau_t^u = 0.
\]

Assuming that we are initially in the steady state ($c_0 = c$ for $t = 0$), where variables without subscript denote steady state values henceforth, we can drop the expressions $\tilde{\beta}^t = \beta^t = 1$, since in intratemporal decisions like the first two equations.
discounting must not appear. The conditions thus become

\[ \lambda_1 e^{-\mu^c} + \lambda_4 e^{-\mu^e} + \lambda_5 h' + \lambda_7 h^T = 0 \] (18)

\[ \lambda_2 + \lambda_3 - \lambda_5 n' - \lambda_7 n^T = 0 \] (19)

\[ \frac{s'c}{\mu^c} + \lambda_1 \left[ 1 + \tau^h - \beta (1 - \delta_h) \right] + \lambda_2 (1 - \tau^n) \sigma^{\mu e} + \lambda_4 mc^{\mu e} \mu^{c e} \left( \beta - \beta \right) + \lambda_5 mh^{c e} \left( \beta - \beta \right) - \lambda_6 \frac{s'c^{\mu e} + 1}{\mu^c} - \lambda_7 \sigma^{\mu e} \left[ g - \tau^h h^T - \tau^n n^T \right] = 0 \] (20)

\[ h^{-\mu^h} \left( 1 - \lambda_1 \frac{\mu^h}{s'h} \right) + \lambda_6 \left[ \beta (1 - \delta_h) - 1 \right] - \lambda_7 \tau^h = 0 \] (21)

\[ -n^{\mu^h} + \lambda_2 \frac{\mu^h (1 - \tau^n)}{s'n} + \lambda_6 - \lambda_7 \tau^n = 0 \] (22)

\[ \frac{(1 - s^l) c'}{\mu^c} + \lambda_3 (1 - \tau^n) c^{\mu e} + \lambda_4 \left[ 1 + \tau^h + m \beta - \beta (1 - \delta_h + m) \right] \]

\[ -\frac{\lambda_5}{\mu^c e^{-\mu^e - 1}} - \frac{\lambda_6 (1 - s^l)}{\mu^c e^{-\mu^e - 1}} = 0 \] (23)

\[ h'^{-\mu^h} \left( 1 - \lambda_4 \frac{\mu^h}{(1 - s^l) h'} \right) + \lambda_5 \left[ \frac{\beta (1 - \delta_h - m) - 1 - \tau^h + m \beta}{1 - s^l} \right] \]

\[ + \lambda_6 \left[ \beta (1 - \delta_h) - 1 \right] - \lambda_7 \tau^h = 0 \] (24)

\[ -n'^{\mu^h} + \lambda_3 \frac{\mu^h (1 - \tau^n)}{(1 - s^l) n'} + \lambda_5 \frac{(1 - \tau^n)}{(1 - s^l)} + \lambda_6 - \lambda_7 \tau^n = 0. \] (25)

The household FOCs and other constraints in needed to determine the steady state
are given by

\[ h^{\mu^b} = c^{-\mu^e} \left[ (1 + \tau^h) - \beta (1 - \delta_h) \right] \] (26)

\[ n^{\mu^e} c^{\mu^e} = (1 - \tau^n) \] (27)

\[ R^g = R = \frac{1}{\beta} \]

\[ h'^{\mu^h} = c^{-\mu^e} \left[ (1 + \tau^h) - \beta' (1 - \delta_h) \right] + m c^{-\mu^e} (\beta - \beta') \]

\[ \Rightarrow h^{\mu^h} = c^{-\mu^e} \left[ (1 + \tau^h) - \beta' (1 - \delta_h) + m (\beta - \beta') \right] \] (28)

\[ n^{\mu^e} c^{\mu^e} = (1 - \tau^n) \] (29)

\[ c' = n' (1 - \tau^n) + h' \left[ m (\beta - 1) - \delta_h - \tau^h \right] \] (30)

\[ g + (1 - \beta) b^g = \tau^h (s'h + (1 - s') h'') + \tau^n (s'n + (1 - s') n') \] (31)

\[ s'c + (1 - s') c' + g = s'n + (1 - s') n' - \delta_h s' h - \delta_h (1 - s') h'. \] (32)
5.5 Alternative Policies

The Lagrangian reads

\[
J = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{l}
\text{s}^t u(c_t, h_t, 1-n_t) + (1 - \text{s}^t) u(c'_t, h'_t, 1-n'_t) \\
+ \lambda_{t,1} \left[ \text{h}_t^{-\mu_h} - \left(1 + \tau_{t}^{h,h} \right) \text{c}_t^{-\mu_c} + \beta \left(1 - \delta_h \right) E_t \text{c}_t^{-\mu_c} \right] \\
+ \lambda_{t,2} \left[ n_t^{\mu_c} \text{c}_t^{\mu_c} - 1 + \tau_{t}^{n_n} \right] + \lambda_{t,3} \left[ n_t^{\mu_c} \text{c}_t^{\mu_c} - 1 + \tau_{t}^{n_n} \right] \\
+ \lambda_{t,4} \left[ \text{h}_t^{-\mu_h} - \left(1 + \tau_{t}^{h,h} \right) \text{c}_t^{-\mu_c} + \beta' \left(1 - \delta_h \right) E_t \text{c}_t^{\mu_c} \right] \\
- m \left( \text{c}_t^{-\mu_c} \text{c}_t^{\mu_c} \beta E_t \text{c}_t^{-\mu_c} - \beta' E_t \text{c}_t^{\mu_c} \right) \\
+ \lambda_{t,5} \left[ -\text{c}_t - \left(1 + \tau_{t}^{h,h} \right) \text{h}_t + (1 - \tau_{t}^{n_n}) n'_t + (1 - \delta_h) h'_t \right] \\
\quad + \frac{m \text{h}_t'}{c_t^{\mu_c}} \beta E_t \text{c}_t^{-\mu_c} - m \text{h}_t' \\
+ \lambda_{t,6} \left[ -s^t \text{c}_t - (1 - s^t) \text{c}_t - g_t - s^t h_t - (1 - s^t) h'_t \right] \\
\quad + s^n_t (1 - s^t) n'_t + (1 - \delta_h) \left(s^n h_{t-1} + (1 - s^t) h'_{t-1}\right) \\
+ \lambda_{t,7} \text{c}_t^{-\mu_c} \left[ g_t - s^t \tau_{t}^{h,h} h_t - (1 - s^t) \tau_{t}^{h,h} h'_t - \tau_{t}^{n_n} \left(s^n_t + (1 - s^t) n'_t\right) \right] + \lambda_{t,7} b_0 \end{array} \right. \nonumber \]
\]

I do not report all the calculations which are same as before but only summarize
the steady state for this new tax system. The steady state conditions are given by

\[
\lambda_1 c^{-\mu} + \lambda_7 s'^h = 0
\]

\[
+ \lambda_4 c'^{-\mu} + \lambda_5 h' + \lambda_7 (1 - s') h' = 0
\]

\[
\lambda_2 + \lambda_3 - \lambda_5 n' - \lambda_7 n^T = 0
\]

\[
\frac{s^c}{\mu^c} + \lambda_1 (\tau^{h,l} + \delta_h) + \lambda_2 (1 - \tau^n) c^{\mu} + \lambda_4 mc^{-\mu} c^{\mu} (1 - \beta)
\]

\[
+ \lambda_5 mh' c^{\mu} (\beta - 1) - \lambda_6 \frac{s^c}{\mu^c} c^{\mu} c^{\mu+1} - \lambda_7 c^{\mu} [g - \tau^{h,l} s^h - \tau^{h,b} (1 - s') h' - \tau^n n^T] = 0
\]

\[
h'^{-\mu} (1 - \lambda_1 \frac{\mu}{s'h}) + \lambda_6 [\beta (1 - \delta_h) - 1] - \lambda_7 \tau^{h,l} = 0
\]

\[
- n'^{\mu} + \lambda_2 \frac{\mu (1 - \tau^n)}{s'h} + \lambda_6 - \lambda_7 \tau^n = 0
\]

\[
\frac{(1 - s') c'}{\mu^c} + \lambda_3 (1 - \tau^n) c^{\mu} + \lambda_4 [m (\beta - 1) + \tau^{h,b} + \delta_h]
\]

\[
- \frac{\lambda_5}{\mu^c c^{\mu-1}} - \frac{\lambda_6 (1 - s') c^{\mu-1}}{\mu^c c^{\mu-1}} = 0
\]

\[
h'^{-\mu} \left( 1 - \lambda_4 \frac{\mu}{(1 - s') h'} \right) + \lambda_5 \left[ \frac{\beta' (1 - \delta_h) - 1 - \tau^{h,b} + m (\beta - \beta')}{(1 - s')} \right]
\]

\[
+ \lambda_6 [\beta' (1 - \delta_h) - 1] - \lambda_7 \tau^b = 0
\]

\[
- n'^{\mu} + \lambda_3 \frac{\mu (1 - \tau^n)}{(1 - s') n'} + \lambda_5 \frac{1 - \tau^n}{(1 - s')} + \lambda_6 - \lambda_7 \tau^n = 0.
\]

The household FOCs and other constraints in needed to determine the steady state.
are given by

\[ h^{-\mu^h} = e^{-\mu^c} \left[ (1 + \tau^{h,l}) - \beta (1 - \delta_h) \right] \]

\[ n^{\mu^h} c^{\mu^c} = (1 - \tau^n) \]

\[ R^g = R = \frac{1}{\beta} \]

\[ h'^{-\mu^h} = e'^{-\mu^c} \left[ (1 + \tau^{h,b}) - \beta' (1 - \delta_h) + m (\beta - \beta') \right] \]

\[ n'^{\mu^h} c'^{\mu^c} = (1 - \tau^n) \]

\[ e' = n' (1 - \tau^n) + h' \left[ m (\beta - 1) - \delta_h - \tau^{h,b} \right] \]

\[ g + (1 - \beta) b^g = \tau^{h,l} s^l h + \tau^{h,b} (1 - s^l) h' + \tau^n (s^l n + (1 - s^l) n') \]

\[ s^l c + (1 - s^l) c' + g = s^l n + (1 - s^l) n' - \delta_h s^l h - \delta_h (1 - s^l) h'. \]
Notes